Lecture 8. Kondo effect in quantum dots

During the elastic cotunnelung process, considered above, the internal state of the dot remains unchanged: the same electron tunnels in and out of the dot. There is, however, a process which becomes important in Coulomb blockade valleys with odd number of electrons, when the ground state contains a single electron on the level n_F (all levels with $n < n_F$ are doubly occupied). This gives a possibility for the incoming and outgoing electrons to be on the same orbital level n_F, but have different spins.

Reduction to the Kondo problem. At low temperatures, the internal state of the dot is just the spin state of the single unpaired electron on the level n_F. Thus, the transport of the electrons through the dot should be described by the following effective Hamiltonian:

$$\hat{H}_{\text{eff}} = \sum_{\alpha k\sigma,\alpha' k'} M_{\alpha' k' \to \alpha k} \hat{c}^\dagger_{\alpha k\sigma} \hat{c}_{\alpha' k'\sigma} + \sum_{\alpha k\sigma,\alpha' k'\sigma'} J_{\alpha\alpha' kk'} \left(\hat{c}^\dagger_{\alpha k\sigma} \frac{\sigma^i_{\sigma\sigma'}}{2} \hat{c}_{\alpha' k'\sigma'} \right) \hat{S}_i, \tag{8.1}$$

where the first term corresponds to the elastic cotunneling, and the second term uses the $SU(2)$ symmetry which should hold in the absence of spin-orbit interactions.

The matrix elements $J_{\alpha\alpha' kk'}$ should be found from the second-order perturbation theory in $t_{\alpha kn}$ as we did for the elastic cotunneling. Namely, the matrix element of the transition

$$|i\rangle = \hat{c}^\dagger_{\alpha' k'\uparrow} \hat{c}^\dagger_{n_F\downarrow} |\Psi_0\rangle \quad \to \quad |f\rangle = \hat{c}^\dagger_{\alpha k\downarrow} \hat{c}^\dagger_{n_F\uparrow} |\Psi_0\rangle,$$

which can go via two intermediate states

$$|0\rangle = \hat{c}^\dagger_{\alpha k\downarrow} \hat{c}^\dagger_{\alpha' k'\uparrow} |\Psi_0\rangle, \quad |2\rangle = \hat{c}^\dagger_{n_F\uparrow} \hat{c}^\dagger_{n_F\downarrow} |\Psi_0\rangle,$$

with energies E_h and E_e, respectively, is given by

$$\frac{J_{\alpha\alpha' kk'}}{2} = t_{\alpha k n_F} t^*_{\alpha' k' n_F} \left(\frac{1}{E_e} + \frac{1}{E_h} \right). \tag{8.2}$$

There is another contribution to the matrix element, due to the term $-J_S \hat{\mathbf{S}}^2$, which can flip spins on two different levels. This, we can inject an electron on a level $n > n_F$, or remove an electron from a level $n < n_F$. The intermediate states are singlet and triplet:

$$|0_{n<n_F,S/T}\rangle = \frac{\hat{c}^\dagger_{n_F\downarrow} \hat{c}_{n\downarrow} \mp \hat{c}^\dagger_{n_F\uparrow} \hat{c}_{n\uparrow}}{\sqrt{2}} |\Psi_0\rangle, \quad |2_{n>n_F,S/T}\rangle = \frac{\hat{c}^\dagger_{n\uparrow} \hat{c}^\dagger_{n_F\downarrow} \mp \hat{c}^\dagger_{n\downarrow} \hat{c}^\dagger_{n_F\uparrow}}{\sqrt{2}} |\Psi_0\rangle,$$

with energies $E_{h,e} \mp \epsilon_n$ and $E_{h,e} \mp \epsilon_n - 2J_S$, respectively. This gives an additional contribution (**check!**)

$$\frac{J_{\alpha\alpha' kk'}}{2} = -\frac{J_S}{2} \sum_{n>n_F} \frac{t_{\alpha kn} t^*_{\alpha' k' n}}{(E_e + \epsilon_n)^2} - \frac{J_S}{2} \sum_{n<n_F} \frac{t_{\alpha kn} t^*_{\alpha' k' n}}{(E_h - \epsilon_n)^2}. \tag{8.3}$$

This contribution has the opposite sign and relative magnitude $J_S/\delta_1 < 1$, so we will neglect it.

Without loss of generality, we can assume the densities of states in the electrodes to be equal (denoted by ν) and $t_{\alpha k n_F}$ to be k-independent. Indeed, what counts is

$$\gamma_{n_F\alpha} = 2\pi \sum_k |t_{\alpha k n_F}|^2 \delta(\xi_{\alpha k}) \equiv \pi\nu t_\alpha^2,$$

and whether it is due to a few well coupled levels or many badly coupled levels, should not matter. So, we define $t_\alpha = \sqrt{\gamma_{n_F\alpha}/(\pi\nu)}$. Let us rotate the basis

$$\hat{a}_{1k\sigma} = \hat{c}_{Rk\sigma}\cos\theta + \hat{c}_{Lk\sigma}\sin\theta, \quad \hat{a}_{2k\sigma} = -\hat{c}_{Rk\sigma}\sin\theta + \hat{c}_{Lk\sigma}\cos\theta, \quad \theta = \arctan\frac{t_L}{t_R}, \tag{8.4}$$

then the Hamiltonian takes the form

$$\hat{H} = \sum_{k\sigma} \xi_k \left(\hat{a}^\dagger_{1k\sigma}\hat{a}_{1k\sigma} + \hat{a}^\dagger_{2k\sigma}\hat{a}_{2k\sigma} \right) + J\hat{S}_i \sum_{k\sigma,k'\sigma'} \left(\hat{a}^\dagger_{1k\sigma} \frac{\sigma^i_{\sigma\sigma'}}{2} \hat{a}_{1k'\sigma'} \right). \tag{8.5}$$

The sector $2k\sigma$ is decoupled from the spin, which corresponds to the one-channel Kondo problem with the dimensionless coupling constant

$$\nu J = \frac{\gamma_L + \gamma_R}{\pi} \left(\frac{2}{E_e} + \frac{2}{E_h} \right) \ll 1. \tag{8.6}$$

Had we not neglected the corrections $\sim J_S/\delta_1$, we would obtain a two-channel Kondo problem. Still, even then the weaker channel asymptotically decouples from the rest.

Linear response conductance. The transferred charge is given by

$$\begin{aligned} \hat{Q} &= \frac{e}{2}\sum_{k\sigma}\left(\hat{c}^\dagger_{Rk\sigma}\hat{c}_{Rk\sigma} - \hat{c}^\dagger_{Lk\sigma}\hat{c}_{Lk\sigma} \right) = \\ &= \frac{e}{2}\sum_{k\sigma}\left[\left(\hat{a}^\dagger_{1k\sigma}\hat{a}_{1k\sigma} - \hat{a}^\dagger_{2k\sigma}\hat{a}_{2k\sigma} \right)\cos 2\theta - \left(\hat{a}^\dagger_{1k\sigma}\hat{a}_{2k\sigma} + \hat{a}^\dagger_{2k\sigma}\hat{a}_{1k\sigma} \right)\sin 2\theta \right]. \end{aligned} \tag{8.7}$$

The conductance G is determined by the response of the current $d\hat{Q}/dt$ to the voltage V, which enters the Hamiltonian as $-V\hat{Q}$ (the electrostatic potential of the left/right electrode is raised/lowered by $V/2$). In the Matsubara representation,

$$\begin{aligned} G(i\omega_m) &= \frac{\omega_m}{2}\int_{-\beta}^{\beta} \left\langle \mathrm{T}_\tau\, e^{\tau\hat{H}}\hat{Q}e^{-\tau\hat{H}}\hat{Q} \right\rangle e^{i\omega_m\tau}\, d\tau = \\ &= -\frac{\omega_m e^2\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\, T\sum_{\epsilon_n}\sum_{k\sigma}\left[\mathcal{G}_{k\sigma,k\sigma}(i\epsilon_n)\, \mathcal{G}^{(0)}_{k\sigma}(i\epsilon_n + i\omega_m) + (\omega_m \to -\omega_m) \right]. = \\ &= \frac{i\omega_m e^2\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\sum_{k\sigma}\oint \frac{dz}{4\pi}\,\mathrm{th}\,\frac{z}{2T}\mathcal{G}_{k\sigma,k\sigma}(z)\left[\mathcal{G}^{(0)}_{k\sigma}(z+i\omega_m) + \mathcal{G}^{(0)}_{k\sigma}(z-i\omega_m) \right] \end{aligned} \tag{8.8}$$

where we used the fact that the $2k\sigma$ particles remain free. Analytical continuation gives

$$\begin{aligned} G(\omega) &= i\omega e^2\,\frac{4\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\sum_{k\sigma}\int\frac{d\epsilon}{2\pi}\frac{d\epsilon'}{2\pi}\left[f_0(\epsilon) - f_0(\epsilon')\right]\mathrm{Im}\,\mathcal{G}_{k\sigma,k\sigma}(\epsilon')\,\mathrm{Im}\,\mathcal{G}^{(0)}_{k\sigma}(\epsilon)\times \\ &\quad\times\left[\frac{1}{\omega - \epsilon' + \epsilon + io} - \frac{1}{\omega + \epsilon' - \epsilon + io} \right], \end{aligned} \tag{8.9}$$

$$\begin{aligned} \mathrm{Re}\,G(\omega) &= \omega e^2\,\frac{4\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\sum_{k\sigma}\int\frac{d\epsilon}{2\pi}\left[f_0(\epsilon) - f_0(\epsilon+\omega)\right]\times \\ &\quad\times\frac{1}{2}\left[\mathrm{Im}\,\mathcal{G}_{k\sigma,k\sigma}(\epsilon+\omega)\,\mathrm{Im}\,\mathcal{G}^{(0)}_{k\sigma}(\epsilon) + \mathrm{Im}\,\mathcal{G}_{k\sigma,k\sigma}(\epsilon)\,\mathrm{Im}\,\mathcal{G}^{(0)}_{k\sigma}(\epsilon+\omega) \right]. \end{aligned} \tag{8.10}$$

Perturbation theory. To the second order in J, we have

$$\mathcal{G}_{k\sigma,k\sigma}(\tau-\tau') = \mathcal{G}^{(0)}_{k\sigma}(\tau-\tau') + J\int_0^\beta d\tau_1 \left\langle \hat{S}_i(\tau_1)\right\rangle \mathcal{G}^{(0)}_{k\sigma}(\tau-\tau_1)\,\frac{\sigma^i_{\sigma\sigma}}{2}\,\mathcal{G}^{(0)}_{k\sigma}(\tau_1-\tau') +$$
$$+ J^2\int_0^\beta d\tau_1\, d\tau_2 \left\langle \mathrm{T}_\tau\, \hat{S}_i(\tau_1)\, \hat{S}_j(\tau_2)\right\rangle \times$$
$$\times\, \mathcal{G}^{(0)}_{k\sigma}(\tau-\tau_1)\sum_{k'\sigma'}\frac{\sigma^i_{\sigma\sigma'}}{2}\,\mathcal{G}^{(0)}_{k'\sigma'}(\tau_1-\tau_2)\,\frac{\sigma^j_{\sigma'\sigma}}{2}\,\mathcal{G}^{(0)}_{k\sigma}(\tau_2-\tau'). \quad (8.11)$$

The operators are taken in the interaction representation, so $\hat{S}_i(\tau) = \hat{S}_i$, as the spin has no dynamics of its own. This gives

$$\left\langle \hat{S}_i(\tau_1)\right\rangle = 0, \quad \left\langle \mathrm{T}_\tau\, \hat{S}_i(\tau_1)\, \hat{S}_j(\tau_2)\right\rangle = \begin{cases} \langle \hat{S}_i\hat{S}_j\rangle, & \tau_1>\tau_2, \\ \langle \hat{S}_j\hat{S}_i\rangle, & \tau_2>\tau_1, \end{cases} = \frac{\delta_{ij}}{4}, \quad (8.12)$$

$$\mathcal{G}_{k\sigma,k\sigma}(i\epsilon_n) = \frac{1}{i\epsilon_n-\xi_k} + \frac{1}{(i\epsilon_n-\xi_k)^2}\,\frac{3J^2}{16}\sum_{k'}\frac{1}{i\epsilon_n-\xi_{k'}} =$$
$$= \frac{1}{i\epsilon_n-\xi_k} + \frac{1}{(i\epsilon_n-\xi_k)^2}\,\frac{3J^2}{16}\,\frac{\nu}{2}\,(-i\pi)\,\mathrm{sign}\,\epsilon_n, \quad (8.13)$$

$$G = \frac{3}{32}\,\frac{e^2}{2\pi}\,\frac{4\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\,(\pi\nu J)^2. \quad (8.14)$$

In the nth order of the perturbation theory we have $\langle \mathrm{T}_\tau \hat{S}_{i_1}(\tau_1)\ldots\hat{S}_{i_n}(\tau_n)\rangle$, for which there is no Wick theorem: $\langle \hat{S}_j\hat{S}_k\hat{S}_l\rangle = (i/8)\, e_{jkl}$, etc. Even though $\hat{S}_i(\tau)$ does not depend on τ, the ordering of $\hat{S}$'s does. Thus, for $n>2$ the spin averages depend on τ_i's, so some frequency is transferred between different electronic Green's functions. We are interested now in the J^4 contribution, so there are $4! = 24$ possible orderings. Rather than struggling with them, we represent the spin operators in terms of fermionic ones:

$$\hat{S}_x = \frac{1}{2}\left(\hat{d}^\dagger_\uparrow\hat{d}_\downarrow + \hat{d}^\dagger_\downarrow\hat{d}_\uparrow\right), \quad \hat{S}_y = -\frac{i}{2}\left(\hat{d}^\dagger_\uparrow\hat{d}_\downarrow - \hat{d}^\dagger_\downarrow\hat{d}_\uparrow\right), \quad \hat{S}_z = \frac{1}{2}\left(\hat{d}^\dagger_\uparrow\hat{d}_\uparrow - \hat{d}^\dagger_\downarrow\hat{d}_\downarrow\right), \quad (8.15)$$

and use the diagrammatic technique for interacting fermions. Let us consider a diagram consisting of a bubble made of the impurity Green's functions attached to the line of the mobile electrons by four interaction lines. As we are interested in $\mathrm{Im}\,\mathcal{G}$, we cut the diagram in the middle. Then $\mathrm{Im}\,\mathcal{G}$ will be determined by the square of the following vertex (let us ignore the spin structure):

$$J^2 T\sum_{\omega_m}\sum_{k'}\frac{1}{i\epsilon_n - i\omega_m - \xi_k}\,\frac{1}{i\epsilon_{n'}+i\omega_m} = \frac{J^2}{2}\sum_{k'}\frac{\mathrm{th}(\xi_{k'}/2T)}{i\epsilon_{n-n'}-\xi_{k'}} = \frac{\nu J^2}{2}\ln\frac{\delta_1}{T}.$$

Here the upper cutoff was taken to be δ_1 (the limit of validity of the Kondo Hamiltonian for the quantum dot), so the expression is valid with logarithmic precision. The conductance in the nth order will be proportional to $(\nu J)^n \ln^{n-2}$, so the perturbation theory is spoiled by the presence of the large logarithm.

Poor man's scaling. First consider an abstract problem whose Hilbert space can be separated in two sectors, with high and low energies, and the Hamiltonian has the corresponding block structure. The Schrödinger equation has the form

$$\begin{pmatrix} H_{\text{high}} & V \\ V^\dagger & H_{\text{low}} \end{pmatrix} \begin{pmatrix} \psi_{\text{high}} \\ \psi_{\text{low}} \end{pmatrix} = E \begin{pmatrix} \psi_{\text{high}} \\ \psi_{\text{low}} \end{pmatrix} \tag{8.16}$$

Eliminating ψ_{high}, we obtain the equation for ψ_{low},

$$H_{\text{eff}}\psi_{\text{low}} = E\psi_{\text{low}}, \quad H_{\text{eff}} = H_{\text{low}} + V^\dagger(E - H_{\text{high}})^{-1}V \approx H_{\text{low}} - V^\dagger H_{\text{high}}^{-1} V. \tag{8.17}$$

For the Kondo problem, let us take the cutoff $E_1 = \delta_1$ and choose $E_2 \ll E_1$, but such that $\nu J \ln(E_1/E_2) \ll 1$, having in mind that we are interested at energies $|E| \ll E_2$. Let the low-energy sector be the one with all states k with $-E_1 < \xi_k < -E_2$ filled and all states with $E_2 < \xi_k < E_1$ empty. The high-energy sector is that which has one electron in the interval $E_2 < \xi_k < E_1$, or one hole with $-E_1 < \xi_k < -E_2$. The part of the Hilbert space containing two or more electrons or holes in these intervals is neglected, because it is invoked in higher orders of perturbation theory, so its contribution will be small in the parameter $\nu J \ln(E_1/E_2) \ll 1$. The Hamiltonian is split as

$$\hat{H} = \hat{H}_> + \hat{H}_< + \hat{V}_e + \hat{V}_h + \hat{V}_e^\dagger + \hat{V}_h^\dagger, \tag{8.18}$$

$$\hat{H}_> = \sum_\sigma \sum_{k:E_2<|\xi_k|<E_1} \xi_k \hat{a}_{k\sigma}^\dagger \hat{a}_{k\sigma}, \tag{8.19}$$

$$\hat{H}_< = \sum_\sigma \sum_{k:|\xi_k|<E_2} \xi_k \hat{a}_{k\sigma}^\dagger \hat{a}_{k\sigma} + J\hat{S}_i \sum_{\sigma\sigma'} \sum_{k,k':|\xi_k|,|\xi_k'|<E_2} \hat{a}_{k\sigma}^\dagger \frac{\sigma_{\sigma\sigma'}^i}{2} \hat{a}_{k'\sigma'}, \tag{8.20}$$

$$\hat{V}_e = J\hat{S}_i \sum_{\sigma\sigma'} \sum_{k:E_2<\xi_k<E_1} \sum_{k':|\xi_k'|<E_2} \hat{a}_{k\sigma}^\dagger \frac{\sigma_{\sigma\sigma'}^i}{2} \hat{a}_{k'\sigma'}, \tag{8.21}$$

$$\hat{V}_h = J\hat{S}_i \sum_{\sigma\sigma'} \sum_{k:|\xi_k|<E_2} \sum_{k':-E_1<\xi_k'<-E_2} \hat{a}_{k\sigma}^\dagger \frac{\sigma_{\sigma\sigma'}^i}{2} \hat{a}_{k'\sigma'}. \tag{8.22}$$

$$\hat{V}_e^\dagger \hat{H}_>^{-1} \hat{V}_e = J^2 \hat{S}_i \hat{S}_j \sum_{|\xi_k|,|\xi_{k'}|<E_2} \sum_{E_2<\xi_{k_{1,2,3}}<E_1} \hat{a}_{k\sigma}^\dagger \frac{\sigma_{\sigma\sigma_1}^i}{2} \hat{a}_{k_1\sigma_1} \frac{\hat{a}_{k_2\sigma_2}^\dagger \hat{a}_{k_2\sigma_2}}{\xi_{k_2}} \hat{a}_{k_3\sigma_3}^\dagger \frac{\sigma_{\sigma_3\sigma'}^j}{2} \hat{a}_{k'\sigma'}$$

$$\hat{V}_h^\dagger \hat{H}_>^{-1} \hat{V}_h = J^2 \hat{S}_i \hat{S}_j \sum_{|\xi_k|,|\xi_{k'}|<E_2} \sum_{E_2<-\xi_{k_{1,2,3}}<E_1} \hat{a}_{k_1\sigma_1}^\dagger \frac{\sigma_{\sigma_1\sigma}^i}{2} \hat{a}_{k\sigma} \frac{\hat{a}_{k_2\sigma_2}^\dagger \hat{a}_{k_2\sigma_2}}{\xi_{k_2}} \hat{a}_{k'\sigma'}^\dagger \frac{\sigma_{\sigma'\sigma_3}^j}{2} \hat{a}_{k_3\sigma_3}$$

Taking into account that $\hat{S}_i \hat{S}_j \sigma^i \sigma^j = -\hat{S}_k \sigma^k$, we obtain the effective Hamiltonian

$$\hat{H}_{\text{eff}} = \sum_\sigma \sum_{k:|\xi_k|<E_2} \xi_k \hat{a}_{k\sigma}^\dagger \hat{a}_{k\sigma} + \left(J + \frac{\nu J^2}{2} \ln\frac{E_1}{E_2}\right) \hat{S}_i \sum_{\sigma\sigma'} \sum_{k,k':|\xi_k|,|\xi_k'|<E_2} \hat{a}_{k\sigma}^\dagger \frac{\sigma_{\sigma\sigma'}^i}{2} \hat{a}_{k'\sigma'}. \tag{8.23}$$

Now we can interpret $J_{\text{eff}} = J + (\nu J^2/2)\ln(E_1/E_2)$, as the new coupling constant and E_2 as the new bandwidth. We continue the process by choosing $E_3 \ll E_2$, and so on. Denoting $\ln(E_i/E_{i+1}) = d\ell$, we obtain

$$\frac{dJ_{\text{eff}}}{d\ell} = \frac{\nu J_{\text{eff}}^2}{2}, \quad J_{\text{eff}}(E) = \frac{J}{1 - (\nu J/2)\ln(\delta_1/E)} = \frac{2/\nu}{\ln(E/T_K)}, \quad T_K \equiv \delta_1 e^{-2/(\nu J)}. \tag{8.24}$$

For $J > 0$, the effective coupling constant J_{eff} grows upon decreasing the energy cutoff. The renormalization flow should be stopped when the cutoff reaches $\max\{T, T_K\}$. Indeed, at $E \sim T$, the eliminated strips $E_{n+1} < E < E_n$ are no longer empty or filled, there is thermal population of electrons and holes. Also, at $E \sim T_K$ (Kondo temperature), the dimensionless coupling constant νJ_{eff} is no longer small, so the perturbative renormalization group is no longer valid. In the temperature interval $T_K \ll T \ll \delta_1$, the conductance can be found with logarithmic precision by simply replacing J in Eq. (8.14) by $\mathrm{J_{eff}}$:

$$G = \frac{3\pi^2}{8}\,\frac{e^2}{2\pi}\,\frac{4\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\,\frac{1}{\ln^2(T/T_K)}. \tag{8.25}$$

For lower temperatures, the problem is in the strong-coupling regime. This means that the ground state is a singlet, where the impurity spin is screened by the conduction electrons, which form the so-called "Kondo cloud".

$\mathcal{T}$-matrix. Even though we cannot sum up all diagrams at energies $E \sim T_K$, their structure is such that the Green's function can be represented as

$$\mathcal{G}_{k\sigma,\,k'\sigma'}(\epsilon) = \delta_{\sigma\sigma'}\mathcal{G}^{(0)}_k(\epsilon) + \mathcal{G}^{(0)}_k(\epsilon)\,\mathcal{T}_{\sigma\sigma'}(\epsilon)\,\mathcal{G}^{(0)}_{k'}(\epsilon). \tag{8.26}$$

It is convenient to work with $\mathcal{T}$-matrix introducing a fictitious coordinate representation,

$$\xi_k = v_F(k-k_F), \quad \nu = 2\sum_k \delta(\xi_k) = 2\int \frac{L\,dk}{2\pi}\,\delta(\xi_k) = \frac{L}{\pi v_F}, \tag{8.27}$$

$$\mathcal{G}_{\sigma\sigma'}(x,x';\epsilon) = \sum_{k,k'} \frac{e^{ikx}}{\sqrt{L}}\,\mathcal{G}_{k\sigma,\,k'\sigma'}(\epsilon)\,\frac{e^{-ik'x'}}{\sqrt{L}} =$$

$$= \frac{1}{iv_F}\,e^{i(k_F+\epsilon/v_F)(x-x')}\left[\theta(x-x')\,\delta_{\sigma\sigma'} + \theta(x)\,\theta(-x')\,\frac{L}{iv_F}\,\mathcal{T}_{\sigma\sigma'}(\epsilon)\right]. \tag{8.28}$$

The particle conservation requires that $\delta_{\sigma\sigma'} - i\pi\nu\mathcal{T}_{\sigma\sigma'}(\epsilon)$ should be a unitary 2×2 matrix $\mathcal{S}_{\sigma\sigma'}(\epsilon)$. Assuming that a low-energy electron cannot flip spin upon scattering (which would require breaking up the singlet and thus paying the binding energy $\sim T_K$), we obtain that the $\mathcal{S}$ should be diagonal, so

$$\mathcal{T}_{\sigma\sigma'}(\epsilon) = \frac{i\delta_{\sigma\sigma'}}{\pi\nu}\left[e^{2i\vartheta_\sigma(\epsilon)} - 1\right]. \tag{8.29}$$

The invariance of the Kondo Hamiltonian under the particle-hole transformation $\hat{a}_{k\uparrow} = \hat{\tilde{a}}^\dagger_{k\downarrow}$, $\hat{a}_{k\downarrow} = -\hat{\tilde{a}}^\dagger_{k\uparrow}$, $\xi_k = -\tilde{\xi}_k$, results in $\mathcal{T}_{\sigma\sigma'}(i\epsilon_n) = -\sigma\sigma'\mathcal{T}_{-\sigma',-\sigma}(-i\epsilon_n)$, which upon analytical continuation on the real axis gives

$$\mathcal{T}_{\sigma\sigma'}(\epsilon) = -\sigma\sigma'\mathcal{T}^*_{-\sigma',-\sigma}(\epsilon) \quad \Rightarrow \quad \vartheta_\uparrow(\epsilon) = -\vartheta_\downarrow(-\epsilon). \tag{8.30}$$

To form a singlet, the Kondo cloud should contain exactly one excess spin 1/2. The number of excess electrons with spin σ around the impurity spin is given by $\vartheta_\sigma(0)/\pi$. This can be seen by considering a train of particles with spacing $2\pi/k_F$ far from the impurity, or by counting the number of states on a ring:

$$kL + 2\vartheta(k) = 2\pi l, \;\; (k+\Delta k)L + 2\vartheta(k+\Delta k) = 2\pi(l+1) \;\Rightarrow\; \frac{1}{\Delta k} = \frac{L}{2\pi} + \frac{1}{\pi}\,\frac{d\vartheta(k)}{dk},$$

$$\int_0^{k_F} \frac{dk}{\Delta k} = \int_0^{k_F}\left[L + 2\,\frac{d\vartheta(k)}{dk}\right]\frac{dk}{2\pi} = \frac{k_F L}{2\pi} + \frac{\vartheta(k_F)}{\pi} \quad \Rightarrow \quad \vartheta_\uparrow(0) - \vartheta_\downarrow(0) = \pi. \tag{8.31}$$

Low-temperature regime. Using representation (8.26), we can rewrite the conductance as

$$\begin{aligned} G &= -\frac{e^2}{2\pi}\frac{4\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\,\pi\sum_{k,\sigma}\left(-\frac{\partial f_0}{\partial \xi_k}\right)\mathrm{Im}\,\mathcal{T}_\sigma(\xi_k) = \\ &= \frac{e^2}{2\pi}\frac{4\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\sum_{\sigma}\int d\xi_k\left(-\frac{\partial f_0}{\partial \xi_k}\right)\sin^2\vartheta_\sigma(\epsilon). \end{aligned} \tag{8.32}$$

At zero temperature, we need only $\delta_\sigma(0) = \pm\pi/2$. At low temperatures, one should expand the phase shifts in energy and occupation numbers (Nozières' local Fermi liquid theory), which gives

$$G = \frac{2e^2}{2\pi}\frac{4\gamma_L\gamma_R}{(\gamma_L+\gamma_R)^2}\left[1-\frac{T^2}{T_K^2}\right]. \tag{8.33}$$