Lecture 7. Cotunneling transport through quantum dots in the Coulomb blockade regime

In the Coulomb blockade valley at low temperatures the first-order processes accounted for by rate equations are exponentially suppressed. Instead, transport is dominated by second-order processes called cotunneling. If we assume N_g not to be close to a half-integer, the total number of electrons in the dot N is given by the nearest integer to N_g. The cost to add (remove) an electron is $E_{e,h} = E_c[1 \pm 2(N - N_g)]$.

Inelastic cotunneling processes are:

$$\hat{c}^\dagger_{\alpha k}\hat{c}_{n'}\hat{c}^\dagger_n\hat{c}_{\alpha' k'}: \qquad E_i = \xi_{\alpha'k'} + \epsilon_{n'} \;\to\; E_{int} = E_e + \epsilon_n + \epsilon_{n'} \;\to\; E_f = \xi_{\alpha k} + \epsilon_n\,,$$
$$\hat{c}^\dagger_n\hat{c}_{\alpha' k'}\hat{c}^\dagger_{\alpha k}\hat{a}_{n'}: \qquad E_i = \xi_{\alpha'k'} + \epsilon_{n'} \;\to\; E_{int} = E_h + \xi_{\alpha k} + \xi_{\alpha'k'} \;\to\; E_f = \xi_{\alpha k} + \epsilon_n\,,$$

with their corresponding initial, intermediate, and final energies. These processes increase the energy of the dot by $\omega_{nn'} = \epsilon_n - \epsilon_{n'}$. The corresponding transition matrix element and the Fermi factor are given by

$$M_{\alpha'k',n'\to\alpha k,n} = -\frac{t_{\alpha k n'}t^*_{\alpha'k'n}}{\xi_{\alpha'k'} - \epsilon_n - E_e} - \frac{t^*_{\alpha'k'n}t_{\alpha k n'}}{\epsilon_{n'} - \xi_{\alpha k} - E_h}, \tag{7.1}$$
$$\mathcal{F}_{\alpha'k',n'\to\alpha k,n} = f_{\alpha'}(\xi_{\alpha'k'})\, f_{n'}(1 - f_n)[1 - f_\alpha(\xi_{\alpha k})]. \tag{7.2}$$

The dot distribution function f_n satisfies the kinetic equation

$$\begin{aligned}\frac{\partial f_n}{\partial t} &= \sum_{\alpha,\alpha'}\sum_{k,k'} 2 \sum_{n'(\neq n)} \left|M_{\alpha'k',n'\to\alpha k,n}\right|^2 \left(\mathcal{F}_{\alpha'k',n'\to\alpha k,n} - \mathcal{F}_{\alpha k,n\to\alpha'k',n'}\right) \times \\ &\quad\times 2\pi\delta(\xi_{\alpha k} + \epsilon_n - \xi_{\alpha'k'} - \epsilon_{n'}) = \\ &= \sum_{\alpha,\alpha',k,k'} 2 \sum_{n'(\neq n)} \int d\omega\, |t_{\alpha k n'}|^2 |t_{\alpha'k'n}|^2\, 2\pi\delta(\epsilon_n - \xi_{\alpha'k'} - \omega)\,\delta(\epsilon_{n'} - \xi_{\alpha'k'} - \omega) \times \\ &\quad\times \{(1 - f_n)f_{n'}[1 - f_\alpha(\xi_{\alpha k})]f_{\alpha'}(\xi_{\alpha'k'}) - f_n(1 - f_{n'})f_\alpha(\xi_{\alpha k})[1 - f_{\alpha'}(\xi_{\alpha'k'})]\} \times \\ &\quad\times \left[\frac{1}{E_h - \omega} + \frac{1}{E_e + \omega}\right]^2. \end{aligned} \tag{7.3}$$

Let us assume $T \gg \delta_1$, then many dot levels participate in the transport, so we can pass to the continuous energy variable ϵ by summation over levels within an energy strip of the width w such that $\delta_1 \ll w \ll T$, which also performs an effective ensemble averaging:

$$\frac{\delta_1}{w}\sum_{n:|\epsilon_n - \epsilon| < w/2} f_n = \bar{f}(\epsilon), \tag{7.4}$$
$$\begin{aligned}\frac{\partial \bar{f}(\epsilon)}{\partial t} = \sum_{\alpha,\alpha'} \frac{\overline{\gamma}_\alpha\overline{\gamma}_{\alpha'}}{\pi\delta_1}\left(\frac{1}{E_e} + \frac{1}{E_h}\right)^2 \int d\epsilon'\, d\omega\, \{[1 - \bar{f}(\epsilon)]\bar{f}(\epsilon')[1 - f_\alpha(\epsilon' - \omega)]f_{\alpha'}(\epsilon - \omega) - \\ - \bar{f}(\epsilon)[1 - \bar{f}(\epsilon')]f_\alpha(\epsilon' - \omega)[1 - f_{\alpha'}(\epsilon - \omega)]\}\,.\end{aligned} \tag{7.5}$$

In the linear response regime we write

$$\bar{f}(\epsilon) = f_0(\epsilon) + \left(-\frac{\partial f_0}{\partial \epsilon}\right)\chi(\epsilon), \quad f_\alpha(\xi) = f_0(\xi) + \left(-\frac{\partial f_0}{\partial \xi}\right)eV_\alpha, \tag{7.6}$$
$$0 = \int \frac{d\epsilon'\, d\omega \sum_{\alpha\alpha'}\overline{\gamma}_\alpha\overline{\gamma}_{\alpha'}[\chi(\epsilon') - \chi(\epsilon) - eV_\alpha + eV_{\alpha'}]}{\mathrm{ch}[\epsilon/(2T)]\,\mathrm{ch}[(\epsilon - \omega)/(2T)]\,\mathrm{ch}[\epsilon'/(2T)]\,\mathrm{ch}[(\epsilon'\omega)/(2T)]} \quad\Rightarrow\quad \chi = 0. \tag{7.7}$$

The current entering the αth contact is given by (the factor of 4 comes from spin):

$$\begin{aligned}
I_\alpha &= \sum_{k,k'} 4e \sum_{n\neq n'} \left|M_{\alpha'k',n'\to\alpha k,n}\right|^2 \left(\mathcal{F}_{\alpha'k',n'\to\alpha k,n} - \mathcal{F}_{\alpha k,n\to\alpha'k',n'}\right) \times \\
&\quad \times 2\pi\delta(\xi_{\alpha k} + \epsilon_n - \xi_{\alpha'k'} - \epsilon_{n'}) = \\
&= 2e\,\frac{\overline{\gamma}_\alpha\overline{\gamma}_{\alpha'}}{\pi\delta_1^2}\left(\frac{1}{E_e}+\frac{1}{E_h}\right)^2 \int d\epsilon'\,d\omega\,d\epsilon\,\{\text{the same combination}\} = \\
&= \frac{\overline{\gamma}_L\overline{\gamma}_R}{16\pi T\delta_1^2}\left(\frac{1}{E_e}+\frac{1}{E_h}\right)^2 \int \frac{2e^2(V_{\alpha'\neq\alpha}-V_\alpha)\,d\epsilon'\,d\omega\,d\epsilon}{\mathrm{ch}[\epsilon/(2T)]\,\mathrm{ch}[(\epsilon-\omega)/(2T)]\,\mathrm{ch}[\epsilon'/(2T)]\,\mathrm{ch}[(\epsilon'-\omega)/(2T)]} \\
&= e^2V\,\frac{\overline{\gamma}_L\overline{\gamma}_R T^2}{\pi\delta_1^2}\left(\frac{1}{E_e}+\frac{1}{E_h}\right)^2 \int \frac{dx\,dy\,dz}{\mathrm{ch}\,x\,\mathrm{ch}(x-y)\,\mathrm{ch}\,z\,\mathrm{ch}(z+y)} = \\
&= e^2V\,\frac{\overline{\gamma}_L\overline{\gamma}_R}{\pi\delta_1^2}\left(\frac{T}{E_e}+\frac{T}{E_h}\right)^2 \int\left(\frac{2y}{\mathrm{sh}\,y}\right)^2 dy = e^2V\,\frac{4\pi}{3}\,\frac{\overline{\gamma}_L\overline{\gamma}_R}{\delta_1^2}\left(\frac{T}{E_e}+\frac{T}{E_h}\right)^2, \qquad (7.8)
\end{aligned}$$

where the integrals are calculated as

$$\int_{-\infty}^{\infty} \frac{dx}{\mathrm{ch}\,x\,\mathrm{ch}(x-y)} = \int_0^{\infty} \frac{2e^y\,dt}{(t+1)(t+e^{2y})} = \frac{2y}{\mathrm{sh}\,y}, \qquad (7.9)$$

$$\int_{-\infty}^{\infty} \frac{y^2\,dy}{\mathrm{sh}^2\,y} = \int_1^{\infty} \frac{\ln^2 t\,dt}{(t-1)^2} = 2\int_1^{\infty} \frac{dt}{t}\ln\frac{t}{t-1} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \zeta(2) = \frac{\pi^2}{6}. \qquad (7.10)$$

Elastic cotunneling processes are those that do not change the internal state of the dot (and thus do not include the spin flip on a single level):

$$\begin{aligned}
&\hat{c}^\dagger_{\alpha k}\hat{c}_n\hat{c}^\dagger_n\hat{c}_{\alpha'k'}: \quad && E_i = \epsilon_{\alpha'k'} \;\to\; E_{int} = E_e + \epsilon_n \;\to\; E_f = \epsilon_{\alpha k}\,, \\
&\hat{c}^\dagger_n\hat{c}_{\alpha'k'}\hat{c}^\dagger_{\alpha k}\hat{c}_n: \quad && E_i = \epsilon_{\alpha'k'} + \epsilon_n \;\to\; E_{int} = E_h + \epsilon_{\alpha k} + \epsilon_{\alpha'k'} \;\to\; E_f = \epsilon_{\alpha k} + \epsilon_n\,,
\end{aligned}$$

and the corresponding transition matrix element is given by

$$M_{\alpha'k'\to\alpha k} = \sum_n t_{\alpha kn}t^*_{\alpha'k'n}\left(\frac{1-f_n}{\xi_{\alpha'k'}-\epsilon_n-E_e} - \frac{f_n}{\epsilon_n-\xi_{\alpha k}-E_h}\right), \qquad (7.11)$$

where $f_n = 0, 1$ is the occupation number of the level n and the minus sign comes from the fermionic commutation relations. The current is given by

$$\begin{aligned}
I_{L\to R} &= 2e\sum_{k,k'}\left[f_L(\xi_{Lk}) - f_R(\xi_{Rk'})\right]\left|M_{L,k\to R,k'}\right|^2 2\pi\delta(\xi_{Lk}-\xi_{Rk'}) = \\
&= \frac{e^2V}{\pi}\int \mathcal{T}(\xi)\left(-\frac{\partial f_0}{\partial\xi}\right)d\xi, \qquad (7.12) \\
\mathcal{T}(\xi) &= 4\pi^2\sum_{k,k'}\left|M_{Lk\to Rk'}\right|^2\delta(\xi_{Lk}-\xi)\,\delta(\xi_{Rk'}-\xi). \qquad (7.13)
\end{aligned}$$

As will be seen shortly, $\mathcal{T}(\xi)$ is a smooth function of ξ varying on a typical scale of $\xi \sim E_C$. At the same time, $-\partial f_0/\partial\xi$ is strongly peaked around zero on the scale $\xi \sim T$. Thus, at low temperatures, the elastic cotunneling kinetic coefficients, appearing in Eq. (6.1), can be simply approximated as

$$G = \frac{e^2}{\pi}\,\mathcal{T}(0). \qquad (7.14)$$

In contrast to the inelastic case, G is not a sum of independent terms over many levels, so it is not a self-averaging quantity, and its statistics should be determined (recall Porter-Thomas distribution of $\gamma_{n\alpha}$). Let us look at it in more detail, recalling Eq. (5.7):

$$\begin{aligned} G &= \frac{e^2}{\pi}\left(\frac{2\pi}{\delta_1}\right)^2 \sum_{k,k',n,n'} \delta(\xi_{Lk})\,\delta(\xi_{Rk'})\,t_{Lkn}t^*_{Rk'n'}t^*_{Lkn'}t_{Rk'n'}\,F(\epsilon_n)\,F(\epsilon_{n'}) = \\ &= \frac{e^2}{\pi\delta_1^2}\sum_k \left[|t_L\psi^*_{Lk}(\mathbf{r}_L)|^2\,2\pi\delta(\xi_{Lk})\right]\sum_{k'}\left[|t_R\psi^*_{Rk'}(\mathbf{r}_R)|^2\,2\pi\delta(\xi_{Rk'})\right] \times \\ &\quad\times \left|\sum_n \phi_n(\mathbf{r}_L)\,\phi^*_n(\mathbf{r}_R)\,F(\epsilon_n)\right|^2 = \\ &= \frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\sum_{n,n'}\tau_n\tau^*_{n'}\,F(\epsilon_n)\,F(\epsilon_{n'}), \end{aligned} \tag{7.15}$$

where we denoted

$$F(\epsilon) = \frac{\delta_1\,\theta(\epsilon)}{E_e+\epsilon} - \frac{\delta_1\,\theta(-\epsilon)}{E_h-\epsilon}, \quad \tau_n = L^d\,\phi_n(\mathbf{r}_L)\,\phi^*_n(\mathbf{r}_R). \tag{7.16}$$

Using the fact that wave functions are real Gaussian random variables with the correlator

$$L^d\,\overline{\phi_n(\mathbf{r}_\alpha)\,\phi_{n'}(\mathbf{r}_{\alpha'})} = \delta_{nn'}\delta_{\alpha\alpha'} \tag{7.17}$$

[see Eq. (5.8)], we can calculate the average

$$\begin{aligned} \overline{G} &= \frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\sum_{n,n'} L^{2d}\,\overline{\phi_n(\mathbf{r}_L)\,\phi_n(\mathbf{r}_R)\,\phi_{n'}(\mathbf{r}_L)\,\phi_{n'}(\mathbf{r}_R)}\,F(\epsilon_n)\,F(\epsilon_{n'}) = \\ &= \frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\sum_n F_n^2(\epsilon_n) \approx \frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\int_{-\infty}^{\infty}\frac{d\epsilon}{\delta_1}\,F^2(\epsilon) = \frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\left(\frac{\delta_1}{E_e}+\frac{\delta_1}{E_h}\right), \end{aligned} \tag{7.18}$$

and the variance

$$\begin{aligned} \overline{G^2} &= \left(\frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\right)^2 \sum_{n_1\ldots n_4} F(\epsilon_{n_1})\,F(\epsilon_{n_2})\,F(\epsilon_{n_3})\,F(\epsilon_{n_4}) \times \\ &\quad\times L^{4d}\,\overline{\phi_{n_1}(\mathbf{r}_L)\,\phi_{n_2}(\mathbf{r}_L)\,\phi_{n_3}(\mathbf{r}_L)\,\phi_{n_4}(\mathbf{r}_L)\,\phi_{n_1}(\mathbf{r}_R)\,\phi_{n_2}(\mathbf{r}_R)\,\phi_{n_3}(\mathbf{r}_R)\,\phi_{n_4}(\mathbf{r}_R)} = \\ &= \left(\frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\right)^2 \sum_{n_1\ldots n_4} F(\epsilon_{n_1})\,F(\epsilon_{n_2})\,F(\epsilon_{n_3})\,F(\epsilon_{n_4}) \times \\ &\quad\times\left(\delta_{n_1n_2}\delta_{n_3n_4}+\delta_{n_1n_3}\delta_{n_2n_4}+\delta_{n_1n_4}\delta_{n_2n_3}\right)^2 = \\ &= \left(\frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\right)^2\left[3\sum_{n,n'}F^2(\epsilon_n)\,F^2(\epsilon_{n'}) + 6\sum_n F^4(\epsilon_n)\right] \approx \\ &= \left(\frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\right)^2\left[3\left(\frac{\delta_1}{E_e}+\frac{\delta_1}{E_h}\right)^2 + 2\left(\frac{\delta_1^3}{E_e^3}+\frac{\delta_1^3}{E_h^3}\right)\right]. \end{aligned} \tag{7.19}$$

Here we replaced the summation over n by integration. This is possible because the integrand is smooth on the scale $E_C \gg \delta_1$.

As the variance is of the same order as the average, the full statistics should be determined, which we will do by calculating all moments. This calculation is facilitated by the fact that in the average of the product $\tau_{n_1}\ldots\tau_{n_{2p}}$, those pairings are more important, where $\phi_{n_1}(\mathbf{r}_L)\ldots\phi_{n_{2p}}(\mathbf{r}_L)$ are paired exactly in the same way as $\phi_{n_1}(\mathbf{r}_R)\ldots\phi_{n_{2p}}(\mathbf{r}_R)$, as it gives the minimal number of constraints on the indices. Terms with larger number of constraints are smaller by the parameter δ/E_C, as seen from the calculation of $\overline{G^2}$. This is equivalent to treating τ_n's themselves as Gaussian random variables with the correlator $\overline{\tau_n\tau_{n'}}=\delta_{nn'}$. Then, the pth moment can be easily calculated:

$$\overline{G^p}=\left(\frac{e^2}{\pi}\frac{\overline{\gamma_L}\,\overline{\gamma_R}}{\delta_1^2}\right)^p\left(\frac{\delta_1}{E_e}+\frac{\delta_1}{E_h}\right)^p\frac{(2p)!}{2^p p!}\equiv\overline{G}^p\,\frac{(2p)!}{2^p p!},\tag{7.20}$$

where $(2p)!/(2^p p!)$ is the total number of all possible pairings of $\tau_{n_1}\ldots\tau_{n_{2p}}$. Generally, the distribution function $P(\xi)$ of a random variable ξ can be reconstructed from its moments $\overline{\xi^p}$ via the characteristic function $\chi(u)=\overline{e^{-iu\xi}}$:

$$P(x)=\overline{\delta(x-\xi)}=\int\frac{du}{2\pi}\,e^{iux}\,\overline{e^{-iu\xi}}=\int\frac{du}{2\pi}\,e^{iux}\sum_{p=0}^{\infty}\frac{(-iu)^p}{p!}\,\overline{\xi^p}.\tag{7.21}$$

For the conductance we have

$$\begin{aligned}P(G)&=\int\frac{du}{2\pi\overline{G}}\,e^{iuG/\overline{G}}\sum_{p=0}^{\infty}\frac{(-iu)^p}{p!}\frac{\overline{G^p}}{\overline{G}^p}=\int\frac{du}{2\pi\overline{G}}\frac{e^{iuG/\overline{G}}}{\sqrt{1+2iu}}=\\&=\int_1^{\infty}\frac{dv}{2\pi\overline{G}}\frac{e^{-(v/2)G/\overline{G}}}{\sqrt{v-1}}=\frac{e^{-G/(2\overline{G})}}{\sqrt{2\pi\overline{G}G}},\end{aligned}\tag{7.22}$$

where we have used

$$\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2}\,x^n=\frac{1}{\sqrt{1-x}}.\tag{7.23}$$

This again gives Porter-Thomas distribution, which is natural because the conductance is determined by a large number of random amplitudes, which is exactly the assumption behind the Porter-Thomas distribution.

The ratio between the typical elastic and the inelastic conductances is

$$\frac{G_{in}}{G_{el}}\sim\frac{T^2}{E_C\delta_1},\tag{7.24}$$

so the elastic mechanism dominates the conductance at sufficiently low temperatures, $T\ll\sqrt{E_C\delta_1}$. However, the conductance fluctuations, which for the inelastic mechanism are parametrically small, $\sim\sqrt{\delta_1/T}\,G_{in}$, are dominated by the elastic mechanism up to even higher temperatures, $T\sim(E_C^2\delta_1)^{1/3}$.