Lecture 6. Electron collisions in quantum dots, many-body localization

The matrix element of the transition between two-electron states $n\sigma, n_1\sigma_1 \to n'\sigma', n_1'\sigma_1'$ due to pair interaction $U(\mathbf{r},\mathbf{r}')$ is given by

$$\int \frac{\phi_{n'}^*(\mathbf{r})\,\phi_{n_1'}^*(\mathbf{r}_1) - \phi_{n_1'}^*(\mathbf{r})\,\phi_{n'}^*(\mathbf{r}_1)}{\sqrt{2}}\,U(\mathbf{r},\mathbf{r}_1)\,\frac{\phi_n(\mathbf{r})\,\phi_{n_1}(\mathbf{r}_1) - \phi_{n_1}(\mathbf{r})\,\phi_n(\mathbf{r}_1)}{\sqrt{2}}\,d^d\mathbf{r}\,d^d\mathbf{r}_1$$

for $\sigma_1 = \sigma$, and by the simple integral for $\sigma_1 = -\sigma$. To find the rate of change of the average occupation number of a state n, we sum over all possible partner states n_1, and over all possible final states n', n_1' (with a factor 1/2 to avoid double counting). This gives the kinetic equation

$$\frac{\partial f_n}{\partial t} = \sum_{n_1,n',n_1'} \left(2|U_{n'n_1'n_1n}|^2 - \mathrm{Re}\,U_{n_1'n'n_1n}U_{n'n_1'n_1n}^*\right) 2\pi\delta(\epsilon_{n'} + \epsilon_{n_1'} - \epsilon_{n_1} - \epsilon_n) \times$$
$$\times \left[f_{n'}f_{n_1'}(1-f_{n_1})(1-f_n) - f_nf_{n_1}(1-f_{n_1'})(1-f_{n'})\right], \tag{6.1}$$

where the factor of 2 at the direct term comes from spin summation. Strictly speaking, the δ function which stands in a discrete sum has zero probability to be satisfied, so the result is zero unless the δ function is broadened by some mechanism. Let us assume such broadening for the moment. Then, the sum contains many terms, so the matrix elements can be averaged.

The matrix element $U_{n_1n_2n_3n_4}$ with all n_i's different has zero average itself, but not its square. For a short-range interaction, $(\delta_1L^d/2)\,\delta(\mathbf{r}-\mathbf{r}')$, the average of the direct term (coinciding with the exchange one, thereby canceling the factor of 2) is given by

$$\overline{|U_{n_1n_2n_3n_4}|^2} = \frac{\delta_1^2}{4L^d}\int d^2\mathbf{r}\,\mathcal{K}_d^4(p_Fr) = \frac{\delta_1^2}{8\pi(p_FL)^d}\begin{cases} 1.54\ldots, & d=2,\\ 1/4, & d=3.\end{cases} \tag{6.2}$$

This coincides with Eq. (36) of Ya. M. Blanter, Phys. Rev. B **54**, 12807 (1996). For the screened Coulomb potential, $\tilde{U}(\mathbf{r}-\mathbf{r}')$ given by the second term in Eq. (5.21), under the assumption $a_s \gg p_F^{-1}$ (which also suppresses the exchange term), we obtain

$$\overline{|U_{n_1n_2n_3n_4}|^2} = \int \frac{d^2\mathbf{r}_1\ldots d^2\mathbf{r}_4}{L^{4d}}\,\mathcal{K}_d^2(p_F|\mathbf{r}_1-\mathbf{r}_3|)\,\tilde{U}(\mathbf{r}_1-\mathbf{r}_2)\,\mathcal{K}_d^2(p_F|\mathbf{r}_2-\mathbf{r}_4|)\,\tilde{U}(\mathbf{r}_3-\mathbf{r}_4)$$
$$= \left(\frac{\delta_1L^d}{2}\right)^2 \delta_1^4L^d\int\frac{d^d\mathbf{q}\,d^d\mathbf{p}\,d^d\mathbf{p}'}{(2\pi)^{3d}}\,\frac{\delta(\epsilon_\mathbf{p}-\epsilon_F)\,\delta(\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_F)\,\delta(\epsilon_{\mathbf{p}'}-\epsilon_F)\,\delta(\epsilon_{\mathbf{p}'-\mathbf{q}}-\epsilon_F)}{[1+(qa_s)^{d-1}]^2}$$
$$\approx \left(\frac{\delta_1L^d}{2}\right)^2\frac{\delta_1^4L^d}{(\delta_1L^d)^2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\int\frac{d^{d-1}\mathbf{n}}{O_d}\,\frac{d^{d-1}\mathbf{n}'}{O_d}\,\frac{\delta(v_F\mathbf{n}\mathbf{q})\,\delta(v_F\mathbf{n}'\mathbf{q})}{[1+(qa_s)^{d-1}]^2} =$$
$$= \frac{\delta_1^4L^d}{4}\left(\frac{O_{d-1}}{O_d}\right)^2\int\frac{d^d\mathbf{q}}{(2\pi)^d}\,\frac{1}{(v_Fq)^2[1+(qa_s)^{d-1}]^2} = \frac{\delta_1^4L^d}{8\pi^2v_F^2}\begin{cases}\ln(L/a_s), & d=2,\\ \pi/(8a_s), & d=3.\end{cases}$$

This result, $\sim \delta_1^2/[(p_FL)^d(p_Fa_s)^{d-2}]$, coincides with the previous one in $d=2$ (apart from the logaritmic factor). In $d=2$, it can also be expressed as $\sim \delta_1^2/g^2$, $g \sim (p_FL)^{d-1}$. For a diffusive dot, one obtains $\sim \delta_1^2/g^2$ in all dimensions. Thus, we write

$$\overline{2|U_{n'n_1'n_1n}|^2 - \mathrm{Re}\,U_{n_1'n's_1n}U_{n'n_1'n_1n}^*} = \lambda^2\delta_1^2, \quad \lambda \ll 1. \tag{6.3}$$

Let us assume that the occupation numbers depend on energy only, $f_n = f(\epsilon_n)$, that this dependence is smooth on the scale δ_1, and use a notation $f(E) \to f_E$. Then

$$\frac{\partial f_E}{\partial t} = \frac{2\pi\lambda^2}{\delta_1} \int dE_1\, d\omega\, [f_{E+\omega} f_{E_1-\omega}(1-f_{E_1})(1-f_E) - f_E f_{E_1}(1-f_{E_1-\omega})(1-f_{E+\omega})]\,. \tag{6.4}$$

The equilibrium distribution function is the one which nullifies the the collision integral, that is, the Fermi-Dirac distribution:

$$f_0(E) = \frac{1}{e^{E/T}+1}. \tag{6.5}$$

To find the collision rate Γ, we prepare a distribution function of the form $f(E) = f_0(E) + A\delta(E - E_0)$, linearize the collision integral, and see how the amplitude A decreases with time, $A \propto e^{-\Gamma t}$. This gives a typical Fermi-liquid result (we replace $E_0 \to E$):

$$\begin{aligned}
\Gamma(E) &= \frac{2\pi\lambda^2}{\delta_1} \int d\omega\, dE_1\, [f_{E_1}(1-f_{E_1-\omega})(1-f_{E+\omega}) + f_{E+\omega} f_{E_1-\omega}(1-f_{E_1})] = \\
&= \frac{2\pi\lambda^2}{\delta_1} \int d\omega\, \frac{dE_1\, 2\,\mathrm{ch}(E/2T)}{8\,\mathrm{ch}[(E+\omega)/2T]\,\mathrm{ch}[(E_1-\omega)/2T]\,\mathrm{ch}[E_1/2T]} = \\
&= \frac{2\pi\lambda^2}{\delta_1} \int d\omega\, \frac{\omega}{2} \left(\mathrm{cth}\,\frac{\omega}{2T} + \mathrm{th}\,\frac{E-\omega}{2T}\right) = \frac{\pi\lambda^2}{\delta_1}\left(E^2 + \pi^2 T^2\right).
\end{aligned} \tag{6.6}$$

where the E_1 integration is performed using

$$\int_{-\infty}^{\infty} \frac{dx}{\mathrm{ch}\,x\,\mathrm{ch}(x+y)} = \int_{-\infty}^{\infty} \frac{4e^{-y}\, e^{2x}\, dx}{(e^{2x}+1)(e^{2x}+e^{-2y})} = \frac{2y}{\mathrm{sh}\,y}, \tag{6.7}$$

and the ω integration is done as

$$\frac{d}{dx} \int \frac{y}{2} [\mathrm{cth}\,y - \mathrm{th}(x+y)]\, dy = \frac{x}{2} \int \frac{dy}{\mathrm{ch}^2\, y} = x,$$

$$\int \frac{y\, dy}{2\,\mathrm{sh}\,y\,\mathrm{ch}\,y} = \int \frac{y\, dy}{4\,\mathrm{sh}\,y} = \sum_{k=0}^{\infty} \int_0^{\infty} e^{-(2k+1)y'} y'\, dy' = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}.$$

Thus evaluated $\Gamma(E) = -2\,\mathrm{Im}\,\Sigma^R(E)$, the self-energy calculated from two diagrams (bubble and exchange), and thus can be assosiated with the width of the spectral peak corresponding to the level n. The result (6.6) tells us that levels with energies $|\epsilon_n| \ll \delta_1/\lambda$ have spectral peaks which do not overlap, $\Gamma(\epsilon_n) \ll \delta_1$ (provided that also $T \ll \delta_1/\lambda$), while at $|\epsilon_n| > \delta_1/\lambda$ they merge into continuous spectrum. The peaks at $|\epsilon_n| \ll \delta_1/\lambda$ have Lorentzian lineshapes, as the typical scale of energy dependence of $\Gamma(E)$ (the typical transferred energy) is of the order of $T \gg \delta_1 \gg \Gamma$.

Let us now examine the validity of the initial assumption of the broadening of the energy δ function, assuming that this broadening comes from the collisions themselves, and not from external mechanisms. The collision rate for an electron on the level n can be written as a sum over final states which are three-particle combinations:

$$\Gamma_n = \sum_{n_1,n_1',n'} \left(2|U_{nn_1n_1'nn'}|^2 - \mathrm{Re}\, U_{n_1'n'n_1n}U^*_{n'n_1'n_1n}\right) \times \\ \times \frac{2\,\mathrm{ch}(\epsilon_n/2T)}{8\,\mathrm{ch}(\epsilon_{n_1}/2T)\,\mathrm{ch}(\epsilon_{n_1'}/2T)\,\mathrm{ch}(\epsilon_{n'}/2T)}\, 2\pi\delta(\epsilon_{n'} + \epsilon_{n_1'} - \epsilon_{n_1} - \epsilon_n). \qquad (6.8)$$

Let us focus on levels n with typical thermal energies, $\epsilon_n \sim T$. The hyperbolic cosines restrict the summations to the energy strip of width $\sim T$ as well. Suppose the δ function has width Γ. The effective number of terms contributing to the sum can be estimated as

$$\sum_{n_1,n_1',n'} \theta(T - |\epsilon_{n_1}|)\,\theta(T - |\epsilon_{n_1'}|)\,\theta(T - |\epsilon_{n'}|)\,\theta(\Gamma - |\epsilon_{n'} + \epsilon_{n_1'} - \epsilon_{n_1} - \epsilon_n|) \sim \\ \sim \int\limits_{-T}^{T} \frac{d\epsilon_{n_1}}{\delta_1}\,\frac{d\epsilon_{n_1'}}{\delta_1}\,\frac{d\epsilon_{n'}}{\delta_1}\,\Gamma\,\delta(\epsilon_{n'} + \epsilon_{n_1'} - \epsilon_{n_1} - \epsilon_n) \sim \frac{\Gamma T^2}{\delta_1^3} \equiv \frac{\Gamma}{\delta_3}, \qquad (6.9)$$

where δ_3 is the three-particle level spacing, which determines the density of final states $1/\delta_3$ entering the Fermi Golden Rule, $\Gamma \sim (\lambda\delta_1)^2/\delta_3$. The condition $\Gamma \gg \delta_3$, which is the self-consistency condition for the Fermi Golden Rule, breaks down at $T < \delta_1/\sqrt{\lambda}$.

To understand what is going on at $T < \delta_1/\sqrt{\lambda}$, let us note that Eq. (6.8) represents an attempt to describe the decay of an initial state, a single electron on the level n with the energy $E_i = \epsilon_n$, into final states, three-particle excitations, with energies $E_f = \epsilon_{n'} + \epsilon_{n_1'} - \epsilon_{n_1}$. When the final states do not form a good continuum, they provide a perturbative correction to the initial wave function:

$$\hat{c}_n^\dagger|\Psi\rangle + \sum_{n_1,n_1',n'} \frac{U_{n'n_1'n_1n}}{\epsilon_n + \epsilon_{n_1} - \epsilon_{n'} - \epsilon_{n_1'}}\,\hat{c}_{n'}^\dagger\hat{c}_{n_1'}^\dagger\hat{c}_{n_1}|\Psi\rangle,$$

where $|\Psi\rangle$ is some many-body state, given by occupations (0 or 1) of all single-particle levels. If all denominators are larger than the matrix element in the numerator, the correction is small. If one or several denominators are small or of the order of the coupling matrix element (which we call a resonance), the single-particle excitation is well mixed with the corresponding three-particle excitations (the weights of the states in the exact eigenstate are of the same order). The three-particle energies are random quantities, the typical spacing between them is δ_3, and the typical value of the matrix element is $\lambda\delta_1$, so when $\lambda\delta_1 > \delta_3$, which is precisely $T > \delta_1/\sqrt{\lambda}$, the probability to find a resonance approaches unity. Then, on the next step of the perturbation theory, three-particle excitations are mixed with five-particle excitations whose level spacing is

$$\frac{1}{\delta_5} = \int\limits_{-T}^{T} \frac{d\epsilon_{n_1}}{\delta_1}\,\frac{d\epsilon_{n_2}}{\delta_1}\,\frac{d\epsilon_{n_1'}}{\delta_1}\,\frac{d\epsilon_{n_2'}}{\delta_1}\,\frac{d\epsilon_{n'}}{\delta_1}\,\delta(\epsilon_{n'} + \epsilon_{n_1'} + \epsilon_{n_2'} - \epsilon_{n_2} - \epsilon_{n_1} - \epsilon_n) \sim \frac{T^4}{\delta_1^5}. \qquad (6.10)$$

These, in turn, are coupled to seven-particle excitations, and so on. To see, whether a finite decay rate is generated, one should add an infinitesimal broadening $\eta \to 0^+$, and study coupling between many-particle excitations to all orders. The problem becomes analogous to that of Anderson localization on a lattice with the effective connectivity $K = \delta_{2m+1}/\delta_{2m-1} \sim T^2/\delta_1^2$. The coupling matrix element between neighboring sites is $V \sim \lambda\delta_1$. The effective disorder strength W is such that $\delta_{2m+1} = W/K^m$, which gives $W \sim \delta_1$. This effective Anderson model has a localization-delocalization transition at some critical value of $K = K_c$, determined by

$$\frac{K_c V}{W} \ln \frac{W}{V} \sim 1 \quad \Rightarrow \quad T_c \sim \frac{\delta_1}{\sqrt{\lambda \ln(1/\lambda)}}. \tag{6.11}$$

The absence of decay of a quasiparticle state corresponds to localization in the Fock space of many-particle excitations.

Strictly speaking, Anderson transition may occur only on an infinite lattice, while the many-body Hilbert space of electrons in a quantum dot is finite. To estimate its size, let us fix some total many-particle energy $\mathcal{E} \sim T^2/\delta_1$, and count the number of all many-body states whose energies fall in the interval of width δ_1 around $\mathcal{E}$. The density of many-body states for N non-interacting fermions is

$$\begin{aligned}
\rho_N(\mathcal{E}) &= \sum_{f_n=0,1} \delta_{\sum f_n, N}\, \delta\Big(\sum f_n \epsilon_n - \mathcal{E}\Big) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-iN\phi - i\mathcal{E}t} \sum_{\{f_n\}} \prod_n e^{if_n\phi + if_n\epsilon_n t} = \\
&= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \exp\left[-iN\phi - i\mathcal{E}t + \sum_n \ln\Big(1 + e^{i\phi + i\epsilon_n t}\Big)\right] = \\
&= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \exp\left[\beta(\mathcal{E} - \mu N) - \beta\Omega(\beta\mu, \beta)\right]_{\beta=-it,\ \beta\mu = i\phi} \approx e^{S(\mathcal{E},N)}.
\end{aligned} \tag{6.12}$$

Here the grand-canonical potential is

$$-\beta\Omega(\beta\mu, \beta) = \sum_n \ln\Big(1 + e^{\beta\mu - \beta\epsilon_n}\Big) = \beta\mu N_0 + \frac{2}{\delta_1}\frac{1}{\beta}\left[\frac{(\beta\mu)^2}{2} + \frac{\pi^2}{6}\right], \tag{6.13}$$

and is found from the equations which also determine the saddle point of the integrand

$$\frac{\partial(-\beta\Omega)}{\partial(\beta\mu)} = N = N_0 + \frac{2}{\delta_1}\mu, \quad -\frac{\partial(-\beta\Omega)}{\partial\beta} = \mathcal{E} = \mathcal{E}_0(\mu) + \frac{2}{\delta_1}\frac{\pi^2}{6}\frac{1}{\beta^2}. \tag{6.14}$$

The entropy is defined as $S(\mathcal{E}, N) = \beta\mathcal{E} - \beta\mu N - \beta\Omega$ at the saddle point, which gives $S(\mathcal{E}) = \sqrt{(4\pi^2/6)(2\mathcal{E}/\delta_1)}$. At $N = N_0$, the exponent of the integrand becomes $\beta\mathcal{E} + (2/\beta\delta_1)[(\beta\mu)^2/2 + \pi^2/6]$, from which we obtain the prefactor of $\rho_N(\mathcal{E})$, as well as the largest generation $m_{\max}$, such that $1/\delta_{2m_{\max}+1} \sim \rho_N(\mathcal{E})$

$$\rho_N(\mathcal{E}) = \frac{e^{\sqrt{(4\pi^2/6)(2\mathcal{E}/\delta_1)}}}{4\sqrt{3}\,\mathcal{E}}, \quad m_{\max} + 1 \approx \frac{S(\mathcal{E})}{\ln(\mathcal{E}/\delta_1)} \sim \frac{T/\delta_1}{\ln(T/\delta_1)}. \tag{6.15}$$

Thus, T_c defines a crossover such that at $T > T_c$ the single-particle spectral function represents a Lorentzian envelope stuffed with δ peaks with spacing $\sim 1/\rho_N$, while $T < T_c$ it is just the main $\delta(\epsilon - \epsilon_n)$ with a few satellites from the first few generations.