Lecture 4. Dynamic localization in periodically driven systems

Let us consider a simple model for a quantum dot, assumed to be large enough, so that (i) it contains many electrons, and (ii) the single-electron spectrum is sufficiently dense:

$$\hat{H}_0 = \sum_n \epsilon_n \hat{c}_n^\dagger \hat{c}_n, \tag{4.1}$$

where n labels discrete energy levels. Their positions are assumed to be random, with the average spacing δ_1 (to be precise, the average density of levels is $1/\delta_1$. Here we neglect the spin (which will give just a factor of 2), electron-electron interaction, and any coupling to outside world. Let the dot electrons be subject to an external microwave perturbation:

$$\hat{V}(t) = \sum_{n,n'} V_{nn'} e^{-i\omega t} \hat{c}_n^\dagger \hat{c}_{n'} + \text{h.c.}, \tag{4.2}$$

where the matrix elements are assumed to be random and uncorrelated among themselves and independent from the energy levels:

$$\overline{V_{n_1 n_2}} = 0, \quad \overline{V_{n_1 n_2} V_{n_3 n_4}} = \left(\delta_{n_1 n_3}\delta_{n_2 n_4} + \delta_{n_1 n_4}\delta_{n_2 n_3}\right) \frac{\Gamma \delta_1}{2\pi}. \tag{4.3}$$

Here Γ is a parameter measuring the perturbation strength, whose physical meaning will be clarified shortly. The independence of the correlator on the difference $\epsilon_{n_1} - \epsilon_{n_2}$ is a strong assumption, valid when the electron motion is chaotic. This may be due to random shape of the dot, or to bulk disorder inside the dot (if the corresponding electron mean free path ℓ is smaller than the dot size L). The quntitative condition for this (to be discussed in more detail on the next lecture) is that all energies appearing in the problem (δ_1, ω, Γ, and some of their combinations to be determined later) should be much smaller than the Thouless energy E_T, defined as the inverse time for the electron to reach the dot boundary: $E_\mathrm{T} \sim v_F/L$ for a ballistic dot with $\ell \gg L$, and $E_\mathrm{T} \sim D/L^2$ for a dot with $\ell \ll L$, where the electron motion is diffusive with the diffusion coefficient $D \sim v_F \ell$ (in all cases, v_F is the Fermi velocity). Then, we can restrict our attention to single-electron states whose energies lie within an interval of the order of Thouless energy E_T around the Fermi level, so that E_T is the largest energy scale in the problem. These states can be described by one of the three ensembles of random matrix theory. Here we assume no spin-orbit coupling and external magnetic field, so the wave functions are real, and so are the dipole matrix elements $V_{nn'}$, which gives the two terms in Eq. (4.3).

Electrons under a microwave perturbation can absorb or emit microwave photons with the rate determined by the Fermi Golden Rule:

$$W_{n\to n'} = 2\pi \sum_{n'} |V_{nn'}|^2 \delta(\epsilon_{n'} - \epsilon_n - \omega) \approx 2\pi \int \frac{d\epsilon'}{\delta_1} \frac{\Gamma \delta_1}{2\pi} \delta(\epsilon' - \epsilon_n - \omega) = \Gamma. \tag{4.4}$$

If we interpret Γ as level broadening, the δ function has an effective width Γ. The Golden Rule makes sense when many $\epsilon_{n'}$'s fall into the interval of the width $\sim \Gamma$ around $\epsilon_n + \omega$, that is, when $\delta_1 \ll \Gamma$. This also justifies the replacement of $|V_{nn'}|^2$ by its ensemble average. Also, we need $\Gamma \ll \omega$, then the final states are well distinct from the initial state.

For many electrons, we introduce the distribution function

$$f(\epsilon) = \delta_1 \sum_n f_n \delta(\epsilon_n - \epsilon), \tag{4.5}$$

where the δ function is again assumed to have a width Γ. This $f(\epsilon)$ satisfies the diffusion equation,

$$\frac{\partial f(\epsilon)}{\partial t} = \Gamma \left[f(\epsilon + \omega) + f(\epsilon - \omega) - 2f(\epsilon) \right] \approx \Gamma \omega^2 \frac{\partial^2 f(\epsilon)}{\partial \epsilon^2}. \tag{4.6}$$

The electrons absorb the energy from the microwave at a constant rate, which can be obtained from Eq. (4.6) as

$$\frac{d}{dt} \int_{-\infty}^{\infty} \epsilon \, f(\epsilon) \, \frac{d\epsilon}{\delta_1} = \frac{\Gamma \omega^2}{\delta_1}. \tag{4.7}$$

Note that the explicit form of $f(\epsilon)$ is not needed; the result is entirely determined by the asymptotics $f(\epsilon \to -\infty) = 1$, $f(\epsilon \to \infty) = 0$, valid for any kind of fermionic particles.

Eq. (4.6) suggests an analogy with propagation of a particle in one dimension. This analogy can be formulated quite precisely. Consider the general Hamiltonian of a closed quantum system subject to an external monochromatic perturbation:

$$\hat{H}(t) = \hat{H}_0 + \hat{V} e^{-i\omega t} + \hat{V}^\dagger e^{i\omega t}. \tag{4.8}$$

Here $\hat{H}_0$ and $\hat{V}$ are some matrices. The time-independent part $\hat{H}_0$ determines some energy levels ϵ_n, then $\hat{V}$ is specified by the matrix elements $V_{nn'}$ between these levels. Looking for the wave function on the nth level $\psi_n(t)$ in the form $\psi_n(t) = c_n(t) e^{-i\epsilon_n t}$, one arrives at the non-stationary Schrödinger equation for the coefficients $c_n(t)$:

$$i \frac{dc_n}{dt} = \sum_{n'} \left[V_{nn'} e^{i(\epsilon_n - \epsilon_{n'} - \omega)t} + V^*_{n'n} e^{i(\epsilon_n - \epsilon_{n'} + \omega)t} \right] c_{n'}. \tag{4.9}$$

Let us now consider another system, whose Hamiltonian is time-independent. It is a one-dimensional lattice whose sites are labeled by an integer $s = -\infty, \ldots, \infty$, on each site s there is a grain with electronic levels $\tilde{\epsilon}_{s,n}$, and the levels n, n' of neighboring grains $s, s+1$ are coupled by the matrix element $V_{nn'}$. Representing the wave function on each level as $C_{s,n}(t) e^{-i\tilde{\epsilon}_{s,n} t}$, we obtain the following Schrödinger equation:

$$i \frac{dC_{s,n}}{dt} = \sum_{n'} \left[V_{nn'} e^{i(\tilde{\epsilon}_{s,n} - \tilde{\epsilon}_{s-1,n'})t} C_{s-1,n'} + V^*_{n'n} e^{i(\tilde{\epsilon}_{s,n} - \tilde{\epsilon}_{s+1,n'})t} C_{s+1,n'} \right]. \tag{4.10}$$

Adding up the equations for all s, we see that the sum $\sum_s C_{s,n}$ satisfies the same Eq. (4.9) as c_n, provided that $\tilde{\epsilon}_{s,n} = \epsilon_n - s\omega$. Thus, the problem for a zero-dimensional system under a time-dependent perturbation has been reduced to that for a one-dimensional system with a time-independent Hamiltonian. Note that this mapping is quite general, and no assumptions about the structure of energy levels ϵ_l or matrix elements $V_{ll'}$ has been made.

Any solution of the linear system (4.10) can be expressed in terms of the eigenstates α with energies λ_α and wave functions $X^\alpha_{s,n}$,

$$C_{s,n}(t)\, e^{-i\tilde{\epsilon}_{s,n}t} = \sum_\alpha a_\alpha X^\alpha_{s,n} e^{-i\lambda_\alpha t}. \tag{4.11}$$

If $X_{s,n}$ is an eigenfunction with the eigenvalue λ, then $X_{s+k,n}$ is also an eigenfunction with the eigenvalue $\lambda + k\omega$, for any integer k, so the summation over α can be split into the summation over states α whose energies lie in the interval $|\lambda_\alpha| < \omega/2$ and over the integers k. Thus, any solution of Eq. (4.9) can be represented in the form

$$\begin{aligned} \psi_n(t) &= c_n(t)e^{-i\epsilon_n t} = \sum_s C_{s,n}(t)e^{-i\tilde{\epsilon}_{s,n}t} = \sum_{s,\alpha,k} a_{\alpha,k} X^\alpha_{s+k,n} e^{-i\lambda_\alpha t - i(s+k)\omega t} = \\ &= \sum_{\alpha,k} a_{\alpha,k}\, \chi^\alpha_n(t)\, e^{-i\lambda_\alpha t}, \end{aligned} \tag{4.12}$$

where $\chi^\alpha_n(t)$ are periodic functions of t with the period $2\pi/\omega$. This statement is known as Floquet theorem and is analogous to the Bloch theorem for systems which are periodic in space. The energies λ_α lying in the interval $|\lambda_\alpha| < \omega/2$ are called quasienergies by analogy with quasimomentum. At $V \to 0$ $\lambda_\alpha = \epsilon_n \bmod \omega$. Orthogonality and completeness of the eigenstates of the stationary problem imply the same for the Floquet states at any time:

$$\sum_{s,n} \left[X^\alpha_{s+k,n}\right]^* X^{\alpha'}_{s+k',n} = \delta_{\alpha\alpha'}\delta_{kk'} \quad \Rightarrow \quad \sum_n [\chi^\alpha_n(t)]^*\, \chi^{\alpha'}_n(t) = \delta_{\alpha\alpha'}, \tag{4.13}$$

$$\sum_{\alpha,k} \left[X^\alpha_{s+k,n}\right]^* X^\alpha_{s'+k,n'} = \delta_{ss'}\delta_{nn'} \quad \Rightarrow \quad \sum_\alpha [\chi^\alpha_n(t)]^*\, \chi^\alpha_{n'}(t) = \delta_{nn'}. \tag{4.14}$$

The evolution operator over the period $\hat{U}_{t_0}$ determines the stroboscopic evolution after N periods $T = 2\pi/\omega$:

$$|\psi(t_0 + NT)\rangle = \hat{U}^N_{t_0}|\psi(t_0)\rangle = \left(\sum_\alpha |\chi^\alpha(t_0)\rangle\, e^{-i\lambda_\alpha T}\, \langle\chi^\alpha(t_0)|\right)^N |\psi(t_0)\rangle. \tag{4.15}$$

The effective Floquet Hamiltonian $\hat{H}^{\rm eff}_{t_0}$, defined by $\hat{U}^N_{t_0} = \exp(-iNT\hat{H}^{\rm eff}_{t_0})$, is thus simply

$$\hat{H}^{\rm eff}_{t_0} = \sum_\alpha |\chi^\alpha(t_0)\rangle\, \lambda_\alpha\, \langle\chi^\alpha(t_0)|. \tag{4.16}$$

The physical meaning of the formal correspondence between Eqs. (4.9) and (4.10) is the following. In the time-dependent picture, the periodic perturbation can induce a transition from a state at some energy ϵ to states at energies $\epsilon \pm \omega$, corresponding to absorption/emission of a photon. In the picture with the time-independent perturbation, the latter induces transitions between the neighboring sites of the array. This correspondence admits several straightforward generalizations. (i) If the perturbation contains higher harmonics of the same frequency ($2\omega, 3\omega \ldots$), they correspond to the coupling to next neighboring sites ($s \leftrightarrow s+2$, $s \leftrightarrow s+3$, $\ldots$). (ii) The presence of several incommensurate frequencies $\omega_1, \ldots, \omega_d$ requires a d-dimensional lattice with sites $\mathbf{s} = (s_1, \ldots\, s_d)$, on-site energies $\epsilon_l - s_1\omega_1 - \ldots - s_d\omega_d$, and the matrix element at each frequency determines the coupling along the corresponding dimension.

If we now go back to our model of a microwave-driven quantum dot, the equivalent stationary problem corresponds to replicating the dot into a granular array, where the grains are labeled by s, and the levels in the sth grain are shifted by $s\omega$ with respect to those of the parent grain $s = 0$. The matrix element $V_{nn'}$ couples the levels of neighboring grains. If we initially put one particle on the level n of the grain $s = 0$, the Fermi Golden Rule gives the escape rate Γ to each of the neighboring grains. The particle will perform a random walk, which results in the diffusive propagation with the diffusion coefficient Γ (the unit of length being the intergrain distance). The large-scale properties of the single-particle wave functions in this system can be established from the Thouless criterion. If we consider L grains, the total level spacing is δ_1/L, while the level broadening is determined by the effective Thouless energy Γ/L^2 (not to be confused with the Thouless energy of the original dot), so the Thouless conductance is $\Gamma/(L\delta_1)$. It becomes ~ 1 at $L \sim \Gamma/\delta_1$, which gives the localization length.

This, the eigenstates $X^{\alpha}_{s,n}$ are localized in s over a typical length Γ/δ_1. On each grain, the wave function is concentrated on $\sim \Gamma/\delta_1$ levels around $n_s = \lambda_\alpha/\delta_1 - s\omega/\delta_1$. Thus, the Floquet eigenstates $\chi_n(t) = \sum_s X_{s,n} e^{-is\omega t}$ are localized in n over $\sim \Gamma\omega/\delta_1^2$ levels. Γ/δ_1 is the typical number of photons that an electron can absorb before the absorption stops, the typical time it takes is $\sim (\Gamma/\delta_1)^2/\Gamma$, so the maximal energy the dot can absorb is

$$\frac{\Gamma\omega^2}{\delta_1}\,\frac{(\Gamma/\delta_1)^2}{\Gamma} \sim \frac{\Gamma\omega^2}{\delta_1^2}.$$

It is worth mentioning another instance of dynamic localization in a periodically driven system – the so-called kicked quantum rotor, defined by the Hamiltonian

$$\hat{H} = -\frac{\partial^2}{\partial\theta^2} + K\cos\theta \sum_{N=-\infty}^{\infty} \delta(t - NT). \tag{4.17}$$

This can be viewed as the Hamiltonian of a particle moving on a circle (the position being described by the angle θ), subject to a periodic force in the form of infinitely sharp δ-pulses whose strength is parametrized by the parameter K. For $K \gg 1$, the system absorbs energy at a constant rate, however, after some time this absorption saturates.

The relation between the dynamic localization in the kicked quantum rotor and the driven chaotic quantum dot can be understood in the time-independent picture of Eq. (4.10). For the quantum rotor, the eigenfunctions of $\hat{H}_0$ are $e^{im\theta}$, with energies $\epsilon_m = m^2$ (m being an integer). The time-dependent perturbation in Eq. (4.17) is proportional to $\cos\theta$, so the only non-vanishing matrix elements are between the levels m and $m \pm 1$. However, for the infinitely sharp δ-kicks, all harmonics are present, so the effective inter-site hopping is of infinite range. For the chaotic quantum dot case with a harmonic perturbation, the matrix elements $V_{nn'}$ do not decrease with the distance $|n - n'|$, but only neighboring sites are coupled. This consideration shows that the quantum rotor with δ-kicks and the harmonically driven chaotic quantum dot are in fact complementary to each other.

It is possible to map the quantum kicked rotor on a somewhat simpler effective tight-binding model. Integration of the Schrödinger equation over a kick and between kicks gives

$$\psi(\theta, NT + 0^+) = e^{-iK\cos\theta}\,\psi(\theta, NT - 0^+), \tag{4.18}$$

$$\psi(\theta, (N+1)T - 0^+) = \int \frac{d\theta'}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\theta-\theta') - im^2 T}\,\psi(\theta', NT + 0^+). \tag{4.19}$$

Let us look for the Floquet eigenstates $\chi(\theta, t)$, then the first relation gives

$$\chi(\theta, \pm 0^+) = \left[1 \mp i \tan\frac{K\cos\theta}{2}\right] \bar{\chi}(\theta), \tag{4.20}$$

while the second relation transforms into

$$e^{-i\lambda T} \int \frac{d\theta}{2\pi}\, e^{-im\theta} \left[1 + i\tan\frac{K\cos\theta}{2}\right] \bar{\chi}(\theta) = e^{-im^2 T} \int \frac{d\theta}{2\pi}\, e^{-im\theta} \left[1 - i\tan\frac{K\cos\theta}{2}\right] \bar{\chi}(\theta), \tag{4.21}$$

which can be written as an effective tight-binding problem

$$\bar{\chi}_m \tan\frac{(m^2 - \lambda)T}{2} + \sum_{m'} \tau_{m-m'} \bar{\chi}_{m'} = 0, \quad \tau_l \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, e^{-il\theta} \tan\frac{K\cos\theta}{2}. \tag{4.22}$$

For large m, the diagonal element is quasi-random (with the Cauchy distribution). However, the hopping is long-range. For even l $\tau_l = 0$, for odd $l \gg 1$ we have

$$\begin{aligned} \tau_l &\approx \int_0^{\pi/2} \frac{2\,d\theta}{\pi} \sum_{n=0}^{n_K} \frac{2}{\sqrt{K^2 - (2n+1)^2\pi^2}}\, \frac{\cos l\theta}{\arccos[(2n+1)\pi/K] - \theta} = \\ &= \sum_{n=0}^{n_K} \frac{2\pi}{\sqrt{K^2 - (2n+1)^2\pi^2}}\, \sin\left[|l| \arccos\frac{(2n+1)\pi}{K}\right]. \end{aligned} \tag{4.23}$$

So, from this tight-binding model it is not obvious at all that eigenstates should be localized. Localization can be demonstrated only by mapping on an effective field theory.