Lecture 3. Diagrammatic technique for disorder averaging, weak localization corrections to conductivity

Disorder potential. Consider a single-particle Hamiltonian in d spatial dimensions,

$$H = \frac{\mathbf{p}^2}{2m} - \epsilon_F + V(\mathbf{r}), \tag{3.1}$$

where energies are counted from the Fermi energy ϵ_F, and $V(\mathbf{r})$ is the disorder potential, which is assumed to have the Gaussian probability distribution:

$$dP[V(\mathbf{r})] \propto \exp\left[-\int \frac{V^2(\mathbf{r})}{2W_0}\,d^d\mathbf{r}\right]\mathcal{D}V, \quad \langle V(\mathbf{r})\rangle = 0, \quad \langle V(\mathbf{r})V(\mathbf{r}')\rangle = W_0\delta(\mathbf{r}-\mathbf{r}'). \tag{3.2}$$

This is called Gaussian white-noise potential, as the Fourier spectrum of the correlator,

$$W_{\mathbf{q}} = \int W(\mathbf{r}-\mathbf{r}')\,e^{-i\mathbf{q}(\mathbf{r}-\mathbf{r}')}d^d(\mathbf{r}-\mathbf{r}') = W_0. \tag{3.3}$$

Physically, it corresponds to the limit of an infinite number of infinitely weak impurities, so that a small volume $d^d\mathbf{r}$ contains enough impurities to validate the central limit theorem.

Finally, we assume W_0 to be small:

$$W_0 \ll \epsilon_F^2\lambda_F^d \sim \frac{\epsilon_F}{\nu_0}, \tag{3.4}$$

where $\epsilon_F^2\lambda_F^d$ is the only parameter of the correct dimensionality to make out of ϵ_F and m. Here ν_0 is the density of states at the Fermi level, corresponding to the clean problem:

$$\nu_0 = \int \frac{2\,d^d\mathbf{p}}{(2\pi)^d}\,\delta\left(\frac{p^2}{2m} - \epsilon_F\right). \tag{3.5}$$

The factor of 2 takes into account the spin degeneracy. Later we will see that the condition (3.4) corresponds to $\ell \gg \lambda_F$, where ℓ is the mean free path, or $\epsilon_F\tau \gg 1$.

Green's functions. Let us define the retarded and advanced single-particle Green's functions $G^{R,A}(\mathbf{r},\mathbf{r}',E|V)$ for a given realization of $V(\mathbf{r})$, as the solutions of the equation

$$\left[E \pm i0^+ + \frac{1}{2m}\frac{\partial^2}{\partial\mathbf{r}^2} + \epsilon_F - V(\mathbf{r})\right]G^{R,A}(\mathbf{r},\mathbf{r}',E|V) = \delta(\mathbf{r}-\mathbf{r}'). \tag{3.6}$$

If $\phi_s(\mathbf{r})$ and ξ_s are the exact eigenfunctions and eigenvalues of the Hamiltonian in Eq. (3.1), the Green's functions can also be expressed as

$$G^{R,A}(\mathbf{r},\mathbf{r}',E|V) = \sum_s \frac{\phi_s(\mathbf{r})\,\phi_s^*(\mathbf{r}')}{E - \xi_s \pm i0^+}. \tag{3.7}$$

Why do we need Green's functions? Because (i) some useful observables can be expressed in terms of them, and (ii) they can be calculated using a systematic procedure justified by the condition (3.4).

Let us define the global conductivity $\sigma_{ij}(\omega)$ as the tensor of response of the *space-averaged* current density $\int \mathbf{j}(\mathbf{r})\,d^d\mathbf{r}/L^d$ to the uniform monochromatically oscillating electric field $(i\omega/c)\mathbf{A}e^{-i\omega t}$, introduced via the vector potential $\mathbf{A}e^{-i\omega t}$. This quantity is not always meaningful; it makes sense in the good metallic regime, as can be seen by estimating its fluctuations. Introducing the notation $G^{R-A} = G^R - G^A$, we can express

$$\sigma_{ij}(\omega) = \frac{ine^2}{m\omega}\,\delta_{ij} + \frac{2e^2}{i\omega}\frac{1}{L^d}\sum_{s,s'}\frac{(f_{s'} - f_s)\,\langle s|v_i|s'\rangle\langle s'|v_j|s\rangle}{\omega - \xi_{s'} + \xi_s + i0^+} =$$

$$= \frac{ine^2}{m\omega}\,\delta_{ij} + \frac{2e^2}{i\omega m^2}\frac{1}{L^d}\int d^d\mathbf{r}\,d^d\mathbf{r}'\int\frac{dE}{2\pi}\frac{dE'}{2\pi}\times$$

$$\times\frac{f(E) - f(E')}{\omega - E + E' + i0^+}\,\frac{\partial G^{R-A}(\mathbf{r},\mathbf{r}',E)}{\partial x_i}\,\frac{\partial G^{R-A}(\mathbf{r}',\mathbf{r},E')}{\partial x'_j}. \quad (3.8)$$

Let us introduce $G_0^{R,A}(\mathbf{r}-\mathbf{r}',E)$, the Green's functions of Eq. (3.6) at $V(\mathbf{r}) = 0$:

$$G_0^{R,A}(\mathbf{r}-\mathbf{r}',E) = \int\frac{d^d\mathbf{p}}{(2\pi)^d}\,\frac{e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}}{E - \xi_\mathbf{p} \pm i0^+}, \quad \xi_\mathbf{p} \equiv \frac{p^2}{2m} - \epsilon_F. \quad (3.9)$$

The exact Green's functions can be written as the perturbation series:

$$G^{R,A}(\mathbf{r},\mathbf{r}',E|V) = G_0^{R,A}(\mathbf{r}-\mathbf{r}',E) + \int d^d\mathbf{r}_1\,G_0^{R,A}(\mathbf{r}-\mathbf{r}_1,E)V(\mathbf{r}_1)G_0^{R,A}(\mathbf{r}_1-\mathbf{r}',E) +$$

$$+\int d^d\mathbf{r}_1\,d^d\mathbf{r}_2\,G_0^{R,A}(\mathbf{r}-\mathbf{r}_1,E)\,V(\mathbf{r}_1)\,G_0^{R,A}(\mathbf{r}_1-\mathbf{r}_2,E)\,V(\mathbf{r}_2)\,G_0^{R,A}(\mathbf{r}_2-\mathbf{r}',E) + \ldots, \quad (3.10)$$

each term of which can be averaged over the disorder. The odd-order terms vanish upon averaging, while in the even-order terms all possible pairings of V's should be taken, and the pair correlator $W(\mathbf{r}-\mathbf{r}')$ should be used. It is convenient to represent this graphically by drawing G_0 as a solid line and each VV correlator as a dashed line.

The averaged Green's function depends only on the difference $\mathbf{r}-\mathbf{r}'$, so it is convenient to introduce its Fourier transform

$$\langle G^{R,A}(\mathbf{r},\mathbf{r}',E|V)\rangle = \int\frac{d^d\mathbf{p}}{(2\pi)^d}\,e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}\bar{G}^{R,A}(\mathbf{p},E). \quad (3.11)$$

For $\bar{G}^{R,A}(\mathbf{p},E)$ we have the following series (we omit the subscripts R, A and the argument E for brevity):

$$\bar{G}(\mathbf{p}) = G_0(\mathbf{p}) + G_0(\mathbf{p})\int\frac{d^d\mathbf{p}_1}{(2\pi)^d}\,W_{\mathbf{p}-\mathbf{p}_1}G_0(\mathbf{p}_1)\,G_0(\mathbf{p}) +$$

$$+ G_0(\mathbf{p})\int\frac{d^d\mathbf{p}_1}{(2\pi)^d}\,W_{\mathbf{p}-\mathbf{p}_1}G_0(\mathbf{p}_1)\,G_0(\mathbf{p})\int\frac{d^d\mathbf{p}_2}{(2\pi)^d}\,W_{\mathbf{p}-\mathbf{p}_2}G_0(\mathbf{p}_2)\,G_0(\mathbf{p}) +$$

$$+ G_0(\mathbf{p})\int\frac{d^d\mathbf{p}_1}{(2\pi)^d}\,W_{\mathbf{p}-\mathbf{p}_1}G_0(\mathbf{p}_1)\int\frac{d^d\mathbf{p}_2}{(2\pi)^d}\,W_{\mathbf{p}_1-\mathbf{p}_2}G_0(\mathbf{p}_2)\,G_0(\mathbf{p}_1)\,G_0(\mathbf{p}) +$$

$$+ G_0(\mathbf{p})\int\frac{d^d\mathbf{p}_1}{(2\pi)^d}\frac{d^d\mathbf{p}_2}{(2\pi)^d}\,W_{\mathbf{p}-\mathbf{p}_1}G_0(\mathbf{p}_1)W_{\mathbf{p}_1-\mathbf{p}_2}G_0(\mathbf{p}_2)\,G_0(\mathbf{p}-\mathbf{p}_1+\mathbf{p}_2)\,G_0(\mathbf{p}) +$$

$$+\ldots. \quad (3.12)$$

Out of three fourth-order terms the first one is the most divergent at $E \to \xi_\mathbf{p}$, as in others the divergence is smeared by the momentum integration. In higher orders, the most divergent terms are those which have the largest number of $G_0(\mathbf{p})$ factors. Selecting the most diverging term in each order, we obtain the subseries

$$\bar{G}(\mathbf{p}) = G_0(\mathbf{p}) + G_0(\mathbf{p})\,\Sigma_1(\mathbf{p})\,G_0(\mathbf{p}) + G_0(\mathbf{p})\,\Sigma_1(\mathbf{p})\,G_0(\mathbf{p})\,\Sigma_1(\mathbf{p})\,G_0(\mathbf{p}) + \ldots = \\ = \frac{1}{E - \xi_{\mathbf{p}} - \Sigma_1(\mathbf{p})}, \tag{3.13}$$

where

$$\Sigma_1^{R,A}(\mathbf{p}, E) = \int \frac{d^d\mathbf{p}_1}{(2\pi)^d}\, W_{\mathbf{p}-\mathbf{p}_1} G_0^{R,A}(\mathbf{p}_1, E). \tag{3.14}$$

Let us look at $\mathrm{Re}\,\Sigma_1$ first:

$$\mathrm{Re}\,\Sigma_1^{R,A}(\mathbf{p}, E) = \fint \frac{d^d\mathbf{p}_1}{(2\pi)^d}\, \frac{W_{\mathbf{p}-\mathbf{p}_1}}{E + \epsilon_F - p_1^2/2m}\,. \tag{3.15}$$

The integral is divergent for $d > 1$, so some upper momentum (or short-distance) cutoff should be introduced. It may come from the finite correlation length of the disorder or from finite bandwidth. Let us assume $p_1^2/(2m) < \epsilon_\infty$. For example, consider $d = 2$:

$$\mathrm{Re}\,\Sigma_1^{R,A}(\mathbf{p}, E) = \frac{mW_0}{2\pi} \ln \frac{E + \epsilon_F}{\epsilon_\infty - \epsilon_F - E} = \\ = \frac{mW_0}{2\pi} \left[\ln \frac{\epsilon_F}{\epsilon_\infty - \epsilon_F} + \frac{E}{\epsilon_F} + \frac{E}{\epsilon_\infty - \epsilon_F} + O(E^2) \right]. \tag{3.16}$$

The logarithmic term can be large, but it does not depend on $\mathbf{p}$ or E, so can be absorbed in ϵ_F. The linear in E term, in principle, renormalizes the quasiparticle weight Z, but it is small as E/ϵ_F, so it can be neglected. To conclude, we can ignore $\mathrm{Re}\,\Sigma_1$ at all.

The imaginary part determines the out-scattering time τ for the state $\mathbf{p}$,

$$\mathrm{Im}\,\Sigma_1^{R,A}(\mathbf{p}, E) = \mp \frac{\nu_0 W_0}{2} \int d\xi_1\, \pi\delta(E - \xi_1) = \mp \frac{\pi\nu_0 W_0}{2} \equiv \mp \frac{1}{2\tau} \equiv \mp\gamma, \tag{3.17}$$

analogously to the Fermi Golden Rule. The mean free path $\ell = v_F\tau$. The singularity in $\bar{G}^{R,A}(\mathbf{p}, E \to \xi_{\mathbf{p}})$ is smeared on the scale $|E - \xi_{\mathbf{p}}| \sim \gamma$. At $|E - \xi_{\mathbf{p}}| \sim \gamma$ all terms of the series (3.13) are of the same order, $\sim 1/\gamma$. Moreover, at $|E - \xi_{\mathbf{p}}| \sim \gamma$ there is no reason to neglect other terms in the perturbation series, not included in the subseries (3.13).

Let us look at the last two terms of Eq. (3.12). Note that $G_0(\mathbf{p}_i)$ is peaked at $\xi_{\mathbf{p}_i} \approx E$ (with the precision γ). In the term before the last, the relevant momenta are those with

$$|\mathbf{p}_1| \approx p_F, \quad |\mathbf{p}_2| \approx p_F, \tag{3.18}$$

or more precisely, with $|p_{1,2} - p_F - E/v_F| \sim 1/\ell$. In the last term, the restriction on momenta is stronger:

$$|\mathbf{p}_1| \approx p_F, \quad |\mathbf{p}_2| \approx p_F, \quad |\mathbf{p} - \mathbf{p}_1 + \mathbf{p}_2| \approx p_F. \tag{3.19}$$

Thus, the volume of the effective integration domain in the last term is smaller by a factor $p_F\ell \gg 1$. This is the case for all diagrams with intersecting impurity lines.

The sum of all diagrams with non-intersecting impurity lines satisfies the following self-consistent equation, known as the self-consistent Born approximation:

$$\bar{G}(\mathbf{p}) = G_0(\mathbf{p}) + G_0(\mathbf{p}) \int \frac{d^d\mathbf{p}_1}{(2\pi)^d}\, W_{\mathbf{p}-\mathbf{p}_1}\, \bar{G}(\mathbf{p}_1)\, \bar{G}(\mathbf{p}). \tag{3.20}$$

Denoting the $\mathbf{p}_1$-integral by $\Sigma(\mathbf{p})$, we can rewrite this equation as

$$\Sigma^{R,A}(\mathbf{p},E) = \int \frac{d^d\mathbf{p}_1}{(2\pi)^d}\,\frac{W_{\mathbf{p}-\mathbf{p}_1}}{E-\xi_{\mathbf{p}_1}-\Sigma^{R,A}(\mathbf{p}_1,E)} \quad\Rightarrow\quad \Sigma^{R,A}(\mathbf{p},E) = \Sigma_0 \mp \frac{i}{2\tau}, \tag{3.21}$$

where Σ_0 is a real constant (which can be ingnored, as before), and τ is given by Eq. (3.17). In the coordinate representation, at $|\mathbf{r}-\mathbf{r}'| \gg 1/p_F$, we have

$$\bar{G}^{R,A}(\mathbf{r}-\mathbf{r}',E) = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\,\frac{e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}}{E-\xi_\mathbf{p}\pm i\gamma} \approx A\,e^{\pm i(p_F+E/v_F)|\mathbf{r}-\mathbf{r}'|-|\mathbf{r}-\mathbf{r}'|/(2\ell)}, \tag{3.22}$$

where $A = A_d(p_F|\mathbf{r}-\mathbf{r}'|)$ is a dimensionality-dependent power-law prefactor.

AC conductivity. From Eq. (3.8), using the analytical properties of G^R, G^A, we can perform one energy integration, to obtain

$$\begin{aligned}\sigma_{ij}(\omega) = {}& \frac{ine^2}{m\omega}\,\delta_{ij} + \frac{2e^2}{\omega m^2}\,\frac{1}{L^d}\int d^d\mathbf{r}\,d^d\mathbf{r}'\int\frac{dE}{2\pi}\times\\ &\times\Big\{[f(E+\omega)-f(E)]\,\partial_i G^R(\mathbf{r},\mathbf{r}',E+\omega)\,\partial_j' G^A(\mathbf{r}',\mathbf{r},E) +\\ &+ f(E)\,\partial_i G^R(\mathbf{r},\mathbf{r}',E+\omega)\,\partial_j' G^R(\mathbf{r}',\mathbf{r},E) -\\ &- f(E+\omega)\,\partial_i G^A(\mathbf{r},\mathbf{r}',E+\omega)\,\partial_j' G^A(\mathbf{r}',\mathbf{r},E)\Big\}.\end{aligned} \tag{3.23}$$

The contribution of G^RG^R and G^AG^A terms is determined by states far from the Fermi surface, so it can be evaluated at $T=\omega=0$:

$$\begin{aligned}&-\frac{2e^2}{\omega m^2}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\,\frac{dE}{2\pi}\,p_ip_j\left[\frac{f(E)}{(E-\xi_\mathbf{p}+i0^+)^2}-\frac{f(E)}{(E-\xi_\mathbf{p}-i0^+)^2}\right] =\\ &= \frac{2e^2}{i\omega m^2}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\,p_ip_j\delta(\xi_\mathbf{p}) = -\frac{ie^2}{m\omega}\,\frac{p_F^2}{d}\,\frac{\nu_0}{m}.\end{aligned} \tag{3.24}$$

Noting that

$$n = \frac{2S_d}{(2\pi)^d}\,\frac{p_F^d}{d}, \qquad \frac{p_F^2}{d}\,\frac{\nu_0}{m} = \frac{p_F^2}{d}\,\frac{1}{m}\,\frac{2S_d}{(2\pi)^d}\,\frac{p_F^{d-1}}{v_F} = n,$$

we see that the G^RG^R and G^AG^A contribution exactly cancels the diamagnetic term.

To average the G^RG^A term, we should sum the diffusion ladder. As the integration over momenta in different sections of the ladder is decoupled, and ∂_i, ∂_j' correspond to different momenta, all terms of the ladder vanish, except the zero-order one, where $\partial_i, \partial_j' \to -p_ip_j$. This gives the Drude formula for the ac conductivity:

$$\sigma_{ij}(\omega) = -\frac{2e^2}{m^2\omega}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\,\frac{dE}{2\pi}\,\frac{p_ip_j[f(E+\omega)-f(E)]}{(E+\omega-\xi_\mathbf{p}+i\gamma)(E-\xi_\mathbf{p}-i\gamma)} = \frac{ne^2\tau/m}{1-i\omega\tau}. \tag{3.25}$$

For a momentum-dependent disorder correlator, high-order terms of the ladder series would not vanish. The summation of the ladder series would lead to a replacement of the out-scattering time by the transport time.

Weak localization correction. To study corrections to the Drude conductivity, we should collect terms small as $1/(p_F\ell)$. The first correction to the Drude conductivity (3.25) is given by the loop with two crossing dashed lines, which carry momenta (following the loop) $\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}', \mathbf{p}+\mathbf{p}'-\mathbf{p}_1, \mathbf{p}$. As expected, the requirement $|\mathbf{p}+\mathbf{p}'-\mathbf{p}_1| \approx p_F$ produces a relative smallness $(p_F\ell)^{-1}$. In the next order, there is the fan diagram with momenta $\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}', \mathbf{p}', \mathbf{p}+\mathbf{p}'-\mathbf{p}_1, \mathbf{p}+\mathbf{p}'-\mathbf{p}_2, \mathbf{p}$. Here all momenta can be put on the Fermi surface by requiring $|\mathbf{p}+\mathbf{p}'| \sim \ell^{-1}$, which fixes the direction of $\mathbf{p}'$ and thus, in two dimensions, it results in a smallness $(p_F\ell)^{-1}$. Moreover, by imposing just $|\mathbf{p}+\mathbf{p}'| \sim \ell^{-1}$, we put all momenta close to the Fermi surface in all higher-order fan diagrams. Thus, we have to sum the whole series.

Denoting $\mathbf{q} = \mathbf{p}+\mathbf{p}'$, we can write the sum (in any dimensionality) as

$$W_0^2 \mathcal{C}(\mathbf{q},\omega) = W_0(\mathcal{B}+\mathcal{B}^2+\ldots) = \frac{W_0\mathcal{B}}{1-\mathcal{B}}, \tag{3.26}$$

$$\mathcal{B} = \int \frac{W_0\, d^d\mathbf{p}/(2\pi)^d}{(E+\omega-\xi_{\mathbf{p}}+i\gamma)(E-\xi_{\mathbf{q}-\mathbf{p}}-i\gamma)}. \tag{3.27}$$

For $q \ll p_F$, we approximate $\xi_{\mathbf{q}-\mathbf{p}} = \xi_{\mathbf{p}-\mathbf{q}} \approx \xi_{\mathbf{p}} - v_F\mathbf{n}\mathbf{q}$, where $\mathbf{n}$ is the direction of the momentum $\mathbf{p}$. For integrals, dominated by the vicinity of the Fermi surface, the momentum integration can be split into the angular part and the energy part:

$$\int \frac{d^d\mathbf{p}}{(2\pi)^d} \approx \int \frac{d^{d-1}\mathbf{n}}{S_d} \int_{-\infty}^{\infty} \frac{\nu_0}{2}\, d\xi, \tag{3.28}$$

where S_d is the full solid angle in d dimensions ($S_1 = 2$, $S_2 = 2\pi$, $S_3 = 4\pi$). The density of states ν_0 is taken at the Fermi surface, and the factor $1/2$ appears because we do not have the spin summation here, while ν_0 is for both spins. The ξ-integral can be calculated by residues (note that the two poles lie in the opposite complex half-planes):

$$\mathcal{B}(\mathbf{q},\omega) = \int \frac{d^{d-1}\mathbf{n}}{S_d} \frac{1}{1-i\omega\tau+i\mathbf{n}\mathbf{q}\ell}, \tag{3.29}$$

where we recalled that $2\gamma = 1/\tau = \pi\nu_0 W_0$ and $\ell = v_F\tau$. Let us focus on the limit $\omega\tau \ll 1$, $q\ell \ll 1$ (the diffusive limit). In the opposite (ballistic) limit, we have $|\mathcal{B}| \ll 1$, so there is need to sum the whole ladder at all. In the diffusive limit, we can expand the fraction:

$$\mathcal{B}_E(\mathbf{q},\omega) = \int \frac{d^{d-1}\mathbf{n}}{S_d} \left[1+i\omega\tau - i\mathbf{n}\mathbf{q}\ell - (\mathbf{n}\mathbf{q})^2\ell^2\right] = 1+i\omega\tau - \frac{q^2\ell^2}{d}. \tag{3.30}$$

The result is expressed in terms of the Boltzmann diffusion coefficient $D = v_F^2\tau/d$:

$$\mathcal{C}(\mathbf{q},\omega) = \frac{\pi\nu_0}{-i\omega+Dq^2}. \tag{3.31}$$

The resulting correction to the conductivity is given by

$$\begin{aligned}
\Delta\sigma_{ij}(\omega \ll 1/\tau) = &-\frac{2e^2}{m^2\omega}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{dE}{2\pi} \times \\
&\times \frac{p_i(q_j-p_j)[f(E+\omega)-f(E)]\, W_0^2\, \mathcal{C}(\mathbf{q},\omega)}{(E+\omega-\xi_{\mathbf{p}}+i\gamma)(E+\omega-\xi_{\mathbf{q}-\mathbf{p}}+i\gamma)(E-\xi_{\mathbf{q}-\mathbf{p}}-i\gamma)(E-\xi_{\mathbf{p}}-i\gamma)} \approx \\
\approx &-\frac{e^2\nu_0}{\omega}\frac{\omega}{2\pi}\frac{v_F^2\delta_{ij}}{d}\int_{-\infty}^{\infty}\frac{d\xi}{(\xi^2+\gamma^2)^2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{\mathcal{C}(\mathbf{q},\omega)}{(\pi\nu_0\tau)^2}.
\end{aligned} \tag{3.32}$$

The prefactor is $\sigma_0/(2\pi\tau)$, the first integral is $4\pi\tau^3$, the behavior of the $\mathbf{q}$ integral strongly depends on the dimensionality. Let us set $\omega = 0$, then the integral is divergent at large q for $d = 3$, at small q for $d = 1$, and logarithmic for $d = 2$:

$$\frac{\Delta\sigma(\omega)}{\sigma_0} = -\frac{2}{\pi\nu_0 D}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{q^2} = -\frac{1}{\pi^2\nu_0 D}\begin{cases}\sim L, & d = 1,\\ \ln(L/\ell), & d = 2,\\ \sim 1/\ell, & d = 3.\end{cases} \tag{3.33}$$

The upper cutoff is $q \sim 1/\ell$, since for $q\ell \gg 1$ the ladder is small. The lower cutoff is $q \sim 1/L$, where L is the sample size.

From the first correction we can estimate the localization length in $d = 1, 2$ as the length where the correction becomes of the order of the main term. For $d = 1$, $\nu_0 D \sim \ell$, so $L_{loc} \sim \ell$, as we have already seen. For $d = 2$, $\pi\nu_0 D = g_0 = p_F\ell/2 \gg 1$ is the dimensionless conductance of a square sample, so the localization length is exponentially large:

$$\frac{\sigma}{\sigma_0} = 1 - \frac{1}{\pi g_0}\ln\frac{L}{\ell} = 0 \quad\Rightarrow\quad L_{loc} \sim \ell e^{\pi g_0}. \tag{3.34}$$

Note also that $\Delta\sigma/\sigma_0$ is the first term of the expansion in $1/g$ of $\beta(g)$ from the scaling theory of localization. For $d = 3$, the correction remains small, so a good metal remains good.

The correction to the conductivity, being an interference effect, is sensitive to processes which destroy quantum coherence. If the electron phase is destroyed after a time τ_φ (which is introduced phenomenologically for the moment), the longest loop that is allowed to contribute to the interference correction, is of the size $\sim L_\varphi = \sqrt{D\tau_\varphi}$. Thus, we replace $\ln(L/\ell) \to \ln(L_\varphi/\ell)$. If L_φ is such that $|\Delta\sigma| \ll \sigma_0$, the correction is called weak localization, in contrast to the strong localization which corresponds to $\sigma \propto e^{-L/L_{loc}}$.