## Lecture 2. Transfer matrix technique in one dimension, scaling theory of localization

Let us consider the same tight-binding model as before, but on a one-dimensional lattice. The on-site energies $\epsilon_n$ have some probability distribution $p(\epsilon)$, assuming $\langle\epsilon_n\rangle = 0$ without the loss of generality. The Schrödinger equation,

$$E\psi_n = \epsilon_n\psi_n - (\psi_{n+1} + \psi_{n-1}), \tag{2.1}$$

can be rewritten in the transfer matrix form as

$$\mathbf{x}_{n+1} = \hat{m}_n\mathbf{x}_n, \quad \mathbf{x}_n = \begin{bmatrix} \psi_n \\ \psi_{n-1} \end{bmatrix}, \quad \hat{m}_n = \begin{bmatrix} \epsilon_n - E & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.2}$$

If second-nearest neighbors were involved, we would need a column of four components. The Shrödinger equation can be viewed as the evolution of the vector $\mathbf{x}_n$ as one moves along the chain.

It is more convenient to work in a different basis, in which the evolution of the clean system ($\epsilon_n = 0$) is trivial. Let us introduce $k$ such that $E = -2\cos k$ and formally write

$$\mathbf{x}_n \equiv \begin{bmatrix} \psi_n \\ \psi_{n-1} \end{bmatrix} = \begin{bmatrix} e^{ikn} & e^{-ikn} \\ e^{ik(n-1)} & e^{-ik(n-1)} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \equiv \hat{R}_n\mathbf{a}_n. \tag{2.3}$$

This gives the rotated transfer matrix

$$\hat{\hat{m}}_n = \hat{R}_{n+1}^{-1}\hat{m}_n\hat{R}_n = 1 + \frac{\epsilon_n}{2i\sin k}\begin{bmatrix} 1 & e^{-2ikn} \\ -e^{2ikn} & -1 \end{bmatrix}, \tag{2.4}$$

$$\hat{R}_{n+1}^{-1} = \frac{1}{2i\sin k}\begin{bmatrix} e^{-ikn} & -e^{-ik(n+1)} \\ -e^{ikn} & e^{ik(n+1)} \end{bmatrix}. \tag{2.5}$$

Thus, for $\epsilon_n = 0$ we have $a_n, b_n = \text{const}$, and $\psi_n = ae^{ikn} + be^{-ikn}$. This elucidates the physical meaning of the product matrix

$$\hat{\mathcal{M}} = \hat{\hat{m}}_n\hat{\hat{m}}_{n-1}\ldots\hat{\hat{m}}_2\hat{\hat{m}}_1 = \hat{R}_{n+1}^{-1}\hat{m}_n\hat{m}_{n-1}\ldots\hat{m}_2\hat{m}_1\hat{R}_1. \tag{2.6}$$

Suppose that $\epsilon_n = 0$ for $n \leq 0$ and $n > L$. Then, $\mathcal{M}$ determines the solution of the scattering problem, relating the amplitudes on the left and on the right:

$$\begin{bmatrix} a_R \\ b_R \end{bmatrix} = \hat{\mathcal{M}}\begin{bmatrix} a_L \\ b_L \end{bmatrix}, \quad \begin{bmatrix} t \\ 0 \end{bmatrix} = \hat{\mathcal{M}}\begin{bmatrix} 1 \\ r \end{bmatrix}, \quad \begin{bmatrix} \tilde{r} \\ 1 \end{bmatrix} = \hat{\mathcal{M}}\begin{bmatrix} 0 \\ \tilde{t} \end{bmatrix}, \tag{2.7}$$

$$\hat{\mathcal{M}} = \begin{bmatrix} t - r\tilde{r}/\tilde{t} & \tilde{r}/\tilde{t} \\ -r/\tilde{t} & 1/\tilde{t} \end{bmatrix}. \tag{2.8}$$

Since $\det\hat{m}_n = 1$, $\det\hat{R}_n = 2i\sin k$, we have $\det\hat{\mathcal{M}} = 1$, which fixes $\tilde{t} = t$. Also, $\sigma_x\hat{\hat{m}}_n\sigma_x = \hat{\hat{m}}_n^*$, so $\sigma_x\hat{\mathcal{M}}\sigma_x = \hat{\mathcal{M}}^*$, which fixes $\tilde{r} = -r^*t/t^*$ and $|r|^2 + |t|^2 = 1$.

The matrix

$$\hat{M}_n = \hat{\tilde{m}}_{n-1} \ldots \hat{\tilde{m}}_1 (\hat{\tilde{m}}_{n-1} \ldots \hat{\tilde{m}}_1)^\dagger \tag{2.9}$$

satisfies the constraints

$$\hat{M}_n^\dagger = \hat{M}_n, \quad \mathbf{a}^\dagger \hat{M}_n \mathbf{a} \geq 0 \; \forall \mathbf{a}, \quad \det \hat{M}_n = 1, \quad \hat{M}_n^* = \sigma_x \hat{M}_n \sigma_x. \tag{2.10}$$

Hence, it can be represented as

$$\hat{M}_n = \begin{bmatrix} \operatorname{ch} \eta_n & e^{i\varphi_n} \operatorname{sh} \eta_n \\ e^{-i\varphi_n} \operatorname{sh} \eta_n & \operatorname{ch} \eta_n \end{bmatrix}, \tag{2.11}$$

where $e^{\pm \eta_n}$ are the eigenvalues. The intensity transmission coefficient, $|t|^2 = 1/\operatorname{ch}^2(\eta/2)$.

The change in the matrix $M_n$ during one step is

$$\begin{aligned} \hat{M}_{n+1} - \hat{M}_n &= \hat{\tilde{m}}_n \hat{M}_n \hat{\tilde{m}}_n^\dagger - \hat{M}_n = \\ &= -\frac{\epsilon_n}{\sin k} \begin{bmatrix} \operatorname{sh} \eta_n \sin(2kn + \varphi_n) & i \operatorname{ch} \eta_n e^{-2ikn} + i \operatorname{sh} \eta_n e^{i\varphi_n} \\ -i \operatorname{ch} \eta_n e^{2ikn} - i \operatorname{sh} \eta_n e^{-i\varphi_n} & \operatorname{sh} \eta_n \sin(2kn + \varphi_n) \end{bmatrix} + \\ &+ \frac{\epsilon_n^2}{2\sin^2 k} \left[\operatorname{ch} \eta_n + \operatorname{sh} \eta_n \cos(2kn + \varphi_n)\right] \begin{bmatrix} 1 & -e^{-2ikn} \\ -e^{2ikn} & 1 \end{bmatrix}. \end{aligned} \tag{2.12}$$

So far, all manipulations were exact. Now we make the assumption of weak disorder, $|\epsilon_n| \ll 1$, and expand to $\epsilon_n^2$, denoting $\Phi_n \equiv 2kn + \varphi_n$:

$$\eta_{n+1} = \eta_n - \frac{\epsilon_n}{\sin k} \sin \Phi_n + \frac{\epsilon_n^2 \cos \Phi_n}{2\sin^2 k} \left(\operatorname{cth} \eta_n \cos \Phi_n + 1\right), \tag{2.13}$$

$$\varphi_{n+1} = \varphi_n - \frac{\epsilon_n}{\sin k} \left(1 + \operatorname{cth} \eta_n \cos \Phi_n\right) + \frac{\epsilon_n^2 \sin \Phi_n}{2\sin^2 k} \left[(1 - 2\operatorname{cth}^2 \eta_n) \cos \Phi_n - \operatorname{cth} \eta_n\right], \tag{2.14}$$

We can average over the fast oscillating terms containing $\Phi_n$ when $k$ is incommensurate with $\pi$. Thus, we are interested in $\eta$ only, so let us write

$$\eta_{n+1} = \eta_n - \epsilon_n \frac{\sin \Phi_n}{\sin k} + \epsilon_n^2 F(\eta_n, \Phi_n). \tag{2.15}$$

This mapping determines the recurrence relation for the probability distribution $P_n(\eta)$:

$$P_{n+1}(\eta) = \int p(\epsilon)\, d\epsilon \int_0^{2\pi} \frac{d\Phi}{2\pi} \int P_n(\eta')\, d\eta'\, \delta\!\left(\eta' - \epsilon \frac{\sin \Phi}{\sin k} + \epsilon^2 F(\eta', \Phi) - \eta\right). \tag{2.16}$$

To open the $\delta$-function, we invert the mapping (2.15) to the second order:

$$\eta' = \eta + \epsilon \frac{\sin \Phi}{\sin k} - \epsilon^2 F(\eta, \Phi), \quad \frac{1}{\partial(\arg \delta)/\partial \eta'} = 1 - \epsilon^2 \frac{\partial F(\eta, \Phi)}{\partial \eta},$$

expand $P_n(\eta')$ to the second order, and approximate $P_{n+1}(\eta) - P_n(\eta) \approx \partial P/\partial n$. This gives the Fokker-Planck equation:

$$\frac{4\sin^2 k}{\langle \epsilon^2 \rangle} \frac{\partial P}{\partial n} = \frac{\partial}{\partial \eta} \operatorname{sh} \eta \frac{\partial}{\partial \eta} \frac{P}{\operatorname{sh} \eta}. \tag{2.17}$$

This equation conserves the total probability,

$$\frac{\partial}{\partial n}\int_0^\infty P(\eta)\,d\eta = 0. \tag{2.18}$$

Fixing the integral to be 1, we eliminate the stationary solution $P(\eta) \propto \operatorname{sh}\eta$. Still, this unphysical solution tells us that the distribution has a tendency to go towards large $\eta$. At $\eta \gg 1$, we can approximate $\operatorname{sh}\eta \approx e^\eta/2$, so the equation becomes

$$\frac{4\sin^2 k}{\langle\epsilon^2\rangle}\frac{\partial P}{\partial n} = -\frac{\partial P}{\partial \eta} + \frac{\partial^2 P}{\partial \eta^2}, \tag{2.19}$$

and this we can easily solve:

$$P_n(\eta) = \sqrt{\frac{L_{loc}}{8\pi n}}\exp\left[-\frac{(\eta - 2n/L_{loc})^2}{8n/L_{loc}}\right], \quad L_{loc} \equiv \frac{8\sin^2 k}{\langle\epsilon^2\rangle}. \tag{2.20}$$

This tells us that the typical value of the transmission coefficient $T \sim e^{-2n/L_{loc}}$, so $L_{loc}$ can be associated with the localization length. On the other hand, we can try to calculate the average transmission,

$$\langle T\rangle \approx \int_0^\infty 4e^{-\eta}\,P_n(\eta)\,d\eta = \sqrt{\frac{2L_{loc}}{\pi n}}\int_0^\infty e^{-n/(2L_{loc})-\eta/2-\eta^2 L_{loc}/(8n)}\,d\eta \sim e^{-n/(2L_{loc})}. \tag{2.21}$$

It decays slower than the typical value, as it is dominated by rare realizations with an anomalously large transmission. These realizations have states localized in the center of the wire and equally well coupled to both ends, and the energy of these states is tuned to $E$ with exponential precision. The coefficient in front of the exponential should not be taken seriously, since the integral comes from $\eta \sim 1$ where the Gaussian solution is not applicable. Still, it estimates the total probability contained in the region $\eta \sim 1$.

Let us calculate the mean free path by treating the disorder as perturbation:

$$\frac{1}{\tau} = 2\pi\langle\epsilon^2\rangle\int\frac{dk'}{2\pi}\,\delta(2\cos k - 2\cos k') = \frac{\langle\epsilon^2\rangle}{2|\sin k|}, \quad \ell = \frac{4\sin^2 k}{\langle\epsilon^2\rangle} = \frac{L_{loc}}{2}. \tag{2.22}$$

This means that localization occurs basically after a few scattering events. The assumption of the absence of coherence between different scattering events, crucial for the Boltzmann equation, is totally wrong in the purely one-dimensional case.

If $p(\epsilon) = (1/W)\,\theta(W/2 - |\epsilon|)$, then

$$\langle\epsilon^2\rangle = \frac{W^2}{12}, \quad L_{loc} = \frac{96\sin^2 k}{W^2}.$$

At $k = \pi/2$, where a commensurate anomaly occurs, $L_{loc} \approx 105/W^2$ instead of 96.

**Thouless criterion.** Consider a thick wire with the cross-section area $S \gg \ell^2$, of length $L \gg \ell$, made of a good 3D metal ($p_F \ell \gg 1$, far from the Anderson localization). No matter what the microscopic model for the disorder is (Gaussian, non-Gaussian, white noise, correlated, *etc.*), electron propagation in such a wire is diffusive at distances $\gg \ell$, with some diffusion coefficient $D$ (assumed to be isotropic). The dc conductivity $\sigma$ satisfies the Einstein relation, $\sigma = e^2 \nu_0 D$, which is also model-independent.

Let us recall that electronic motion in the wire is described by the single-particle Schrödinger equation, which gives some energy levels. In the wire of a finite size that we are considering, the levels are discrete, random, and the typical spacing between them is $\delta = 2/(\nu_0 S L)$, where the factor of 2 takes into account the spin degeneracy. If we connect the two ends of the wire to some reservoirs by some good contacts, each electronic state acquires a width since electrons can escape into the reservoirs. The typical width $\Gamma$ is given by the inverse time it takes the electron to diffuse to the end of the wire: $\Gamma \sim D/L^2$, also called Thouless energy. The ratio between the two is called Thouless conductance, $g = 2\pi\Gamma/\delta$. It is indeed the wire conductance, measured in the natural units of $2e^2/(2\pi) \approx (13\,\mathrm{k}\Omega)^{-1}$ (the conductance quantum):

$$\frac{2\pi}{2e^2}\frac{\sigma S}{L} = \frac{2\pi\nu_0 D S}{2L} = \frac{2\pi D/L^2}{2/(\nu_0 S L)} = g. \tag{2.23}$$

Let us now join two pieces of length $L$ into a one piece of length $2L$ and try to guess the nature of the states in the combined piece. The states in the two pieces are coupled by a typical matrix element $\sim D/L^2$ (this is the energy scale by which the electron knows what happens at the boundary). If $D/L^2 \gg \delta$, the states are well coupled and we obtain a good diffusive piece of length $2L$. If $D/L^2 \ll \delta$, the states in each piece retain their individual character (the eigenstates of the whole will be given by eigenstates in each piece with a weak perturbative admixture of the states in the other piece). This means that the states are localized. Thus, the conductance of the wire is given by $[2e^2/(2\pi)]g(L)$ as long as $g(L) \gg 1$, and when $g(L) \sim 1$, the localization occurs. In the localized regime, both the width $\Gamma$ and the conductance decrease exponentially with $L$. This means that the localization length is the one for which $g(L) \sim 1$, which gives $L \sim \nu_0 D S \sim \ell(p_F^2 S)$.

Note that all this occurs in the situation when the disorder is weak, when the Anderson's arguments would tell us that the eigenstates of the system should be extended.

**Single-parameter scaling.** Consider a $d$-dimensional cube of size $L$. Let us assume that the Thouless conductance $g(L)$ is the *only* relevant dimensionless parameter which determines the change of energy levels when the cubes are glued together. Then, $g(2L) = \mathcal{F}(g(L))$, where $\mathcal{F}(g)$ is a universal function of $g$, which depends on the dimensionality, on the symmetries of the problem, but not on microscopic details of the disorder.

For large $L \gg \ell$ it is convenient to pass to the logarithmic variable $\mathcal{L} = \ln(L/\ell)$. Then, when $L \to 2L$, the increment of $\mathcal{L}$, $\mathcal{L} \to \mathcal{L} + \ln 2$ is small, $\ln 2 \ll \mathcal{L}$. The change in $\ln g$ should also be small, so we can use the differential form

$$\frac{d\ln g}{d\mathcal{L}} = \beta(\ln g), \quad \beta(\ln g) = \frac{1}{\ln 2}\,\ln\frac{\mathcal{F}(g)}{g}. \tag{2.24}$$

The initial condition for the differential equation, $g = g_0$ at $\mathcal{L} = 0$, is determined by the disorder strength. The solution at $\mathcal{L} \to \infty$ tells us whether the system is a conductor or an insulator in the thermodynamic limit.

We do not know the function $\beta(\ln g)$, but we can guess its asymptotics from very general arguments. For strong disorder, eigenstates are exponentially localized in any dimensionality, so that $g(L) \propto e^{-\kappa L}$, which means that

$$g(L) \propto e^{-\kappa L} \quad \Rightarrow \quad \beta(\ln g \to -\infty) = \frac{d(-\kappa\ell e^{\ln(L/\ell)})}{d\ln(L/\ell)} = \ln g + \text{const.} \tag{2.25}$$

When $g$ is large, the dynamics is diffusive, so the conductance should follow the usual Ohm's law:

$$g(L) = \frac{\sigma L^{d-2}}{2e^2/(2\pi)} \quad \Rightarrow \quad \beta(\ln g \to \infty) = \frac{d\ln(L/\ell)^{d-2}}{d\ln(L/\ell)} = d-2. \tag{2.26}$$

These two asymptotics should be joined smoothly at finite values of $\ln g$. If we start at some point $\ln g_0$ at $\ln L = 0$, the direction of the flow is determined by the sign of $\beta(\ln g_0)$. Since for $d = 1, 2$ $\beta(\ln g) < 0$ for any $g$, this means that in $d = 1, 2$ the infinite system is always an insulator, even for an arbitrarily weak disorder, so that all eigenstates should be localized.

For $d = 3$, $\beta(\ln g)$ vanishes for some value $g = g_c$. If $g_0 > g_c$, then $g$ increases with $L$, and the system ends up in the Ohmic regime. If $g_0 < g_?$, the system flows towards exponential localization. Thus, the unstable fixed point $g = g_c$ is the point of Anderson transition. Let $s = d\beta(\ln g)/d\ln g$ at $g = g_c$ be the slope of the curve at this point. Then, when we integrate the differential equation (2.24) starting from a value $g_0 \approx g_c$, most of the "time" $\mathcal{L}$ needed to arrive to the Ohmic or exponential regime will be spent in the vicinity of $g_c$ where the linear approximation for $\beta(\ln g)$ is valid. The conductivity $\sigma$ in the Ohmic regime can be estimated as $\sigma \sim (2e^2/2\pi)g_*/L_*$, where $g_* \sim g_c$ and $L_*$ is obtained from the integration:

$$\left(\frac{L_*}{\ell}\right)^s \ln\frac{g_0}{g_c} = \ln\frac{g_*}{g_c} \sim 1 \quad \Rightarrow \quad \sigma \sim \frac{e^2}{\ell}\left(\frac{g_0 - g_c}{g_c}\right)^{1/s}. \tag{2.27}$$

On the insulating side, $L_*$ determines the length needed to enter the exponentially localized regime, and this gives the scaling of the localization length $\propto 1/(g_c - g_0)^{1/s}$.