Lecture 1. Anderson localization in 3D, locator expansion

Consider a tight-binding model on a 3D cubic lattice whose sites are labeled by integers $\mathbf{m} = (m_x, m_y, m_z)$:

$$\hat{H} = \sum_{\mathbf{m}} \epsilon_{\mathbf{m}} |\mathbf{m}\rangle\langle\mathbf{m}| - \sum_{\mathbf{m},\mathbf{m}'} V_{\mathbf{m}\mathbf{m}'} |\mathbf{m}\rangle\langle\mathbf{m}'|, \tag{1.1}$$

where $\epsilon_{\mathbf{m}}$ are random energies uniformly distributed in the range $-W/2 < \epsilon_{\mathbf{m}} < W/2$, and $V_{\mathbf{m}\mathbf{m}'} = 0$ unless $\mathbf{m}, \mathbf{m}'$ are nearest neighbors, in which case it is equal to V. For $W = 0$ the eigenstates are plane waves with the dispersion

$$E_{\mathbf{k}} = -2V(\cos k_x + \cos k_y + \cos k_z) \approx -6V + Vk^2 \;(k \ll 1). \tag{1.2}$$

For weak disorder, the out-scattering rate is given by the Fermi Golden Rule,

$$\begin{aligned} \frac{1}{\tau} &= 2\pi \int \frac{L^3 d^3\mathbf{k}'}{(2\pi)^3} \left| \sum_{\mathbf{m}} \epsilon_{\mathbf{m}} \frac{e^{i(\mathbf{k}-\mathbf{k}')\mathbf{m}}}{L^3} \right|^2 \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) = \\ &= \frac{2\pi}{L^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \sum_{\mathbf{m},\mathbf{m}'} \epsilon_{\mathbf{m}}\epsilon_{\mathbf{m}'}\, e^{i(\mathbf{k}-\mathbf{k}')(\mathbf{m}-\mathbf{m}')}\, \delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) \end{aligned} \tag{1.3}$$

Since different $\epsilon_{\mathbf{m}}$ and $\epsilon_{\mathbf{m}'}$ have random and uncorrelated signs for $\mathbf{m} \neq \mathbf{m}'$, the sum is contributed only by the terms with $\mathbf{m} = \mathbf{m}'$. These terms give

$$\frac{1}{L^3}\sum_{\mathbf{m}} \epsilon_{\mathbf{m}}^2 = \langle \epsilon_{\mathbf{m}}^2 \rangle = \frac{W^2}{12} \quad (L \sim \ell \gg 1), \qquad \frac{1}{\tau} = \frac{2\pi W^2}{12} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \delta(E_{\mathbf{k}} - E'_{\mathbf{k}}). \tag{1.4}$$

In the case $k \ll 1$, this gives

$$\frac{1}{\tau} = \frac{W^2 k}{24\pi V}, \qquad \ell = 2Vk\tau = \frac{48\pi V^2}{W^2}. \tag{1.5}$$

For $E_{\mathbf{k}} = 0$, we have

$$\frac{1}{\tau} = \frac{\pi W^2}{12V} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \left(\int_{-\pi}^{\pi} \frac{dk_x}{2\pi}\, e^{it\cos k_x} \right)^3 = \frac{W^2}{24V} \int_{-\infty}^{\infty} J_0^3(t)\, dt \approx 0.075\, \frac{W^2}{V}, \quad \ell \approx 25 \frac{V^2}{W^2}. \tag{1.6}$$

So, we have problems at low energies when $W \ll V$ and everywhere at $W \sim V$.

To see what is happening at $V \ll W$, we try to analyze the problem from the opposite limiting case, treating the diagonal part of the Hamiltonian, $\epsilon_{\mathbf{m}}$, as the zero approximation, and the hopping term as perturbation. In the zeroth order, the eigenfunctions are localized on one site: for the site $\mathbf{0}$, $\psi_{\mathbf{m}} = \delta_{\mathbf{m}\mathbf{0}}$. In the first order,

$$\psi_{\mathbf{m}} = \delta_{\mathbf{m}\mathbf{0}} + \frac{V_{\mathbf{m}\mathbf{0}}}{\epsilon_{\mathbf{0}} - \epsilon_{\mathbf{m}}}. \tag{1.7}$$

If $V \ll W$, the second term is most likely small, but there is a small probability that it is large, so the eigenstate is well spread on both sites. In higher orders of perturbation theory, required to go to large distances, we encounter a similar situation. So everything is determined by the probability to find resonant $\epsilon_{\mathbf{m}} \approx \epsilon_{\mathbf{0}}$ at large distances.

To formalize the problem, let us introduce the Green's function

$$G_{\mathbf{m}'\mathbf{m}}(E) \equiv \int \frac{dt}{2\pi}\, e^{iEt}\, G_{\mathbf{m}'\mathbf{m}}(t) =$$
$$= -i \lim_{\eta\to 0^+} \int_0^\infty dt\, e^{iEt-\eta t}\, \langle \mathbf{m}'|e^{-i\hat{H}t}|\mathbf{m}\rangle = \langle \mathbf{m}'|(E-\hat{H}+i0^+)^{-1}|\mathbf{m}\rangle, \qquad (1.8)$$

which shows how the state initially localized on the site $\mathbf{m}$ at $t=0$, is spread over other sites $\mathbf{m}'$ at later times t. This Green's function satisfies the Schrödinger equation,

$$E\, G_{\mathbf{m}'\mathbf{m}}(E) = -\lim_{\eta\to 0^+} \int_0^\infty dt\, \frac{de^{iEt-\eta t}}{dt}\, \langle \mathbf{m}'|e^{-i\hat{H}t}|\mathbf{m}\rangle =$$
$$= -\lim_{\eta\to 0^+} e^{iEt-\eta t}\, \langle \mathbf{m}'|e^{-i\hat{H}t}|\mathbf{m}\rangle\Big|_0^\infty - i\lim_{\eta\to 0^+} \int_0^\infty dt\, e^{iEt-\eta t}\, \langle \mathbf{m}'|\hat{H}e^{-i\hat{H}t}|\mathbf{m}\rangle =$$
$$= \delta_{\mathbf{m}'\mathbf{m}} + \epsilon_{\mathbf{m}'} G_{\mathbf{m}'\mathbf{m}}(E) + \sum_{\mathbf{m}_1} V_{\mathbf{m}'\mathbf{m}_1} G_{\mathbf{m}_1\mathbf{m}}(E), \qquad (1.9)$$

whose iterative solution gives

$$G_{\mathbf{m}'\mathbf{m}}(E) = \frac{\delta_{\mathbf{m}'\mathbf{m}}}{E-\epsilon_{\mathbf{m}}} + \frac{V_{\mathbf{m}'\mathbf{m}}}{(E-\epsilon_{\mathbf{m}'})(E-\epsilon_{\mathbf{m}})} + \sum_{\mathbf{m}_1} \frac{V_{\mathbf{m}'\mathbf{m}_1} V_{\mathbf{m}_1\mathbf{m}}}{(E-\epsilon_{\mathbf{m}'})(E-\epsilon_{\mathbf{m}_1})(E-\epsilon_{\mathbf{m}})} + \dots . \qquad (1.10)$$

This is called locator expansion. Indeed, the zero-order Green's function corresponds to the particle localized on one site, in contrast to the Green's function in a translationally invariant system, which describes free propagation and is called propagator.

Let us define the self-energy $\Sigma_{\mathbf{m}}(E)$ by the relation

$$G_{\mathbf{m}\mathbf{m}}(E) = \frac{1}{E-\epsilon_{\mathbf{m}}-\Sigma_{\mathbf{m}}(E)}. \qquad (1.11)$$

In contrast to the usual definition (the sum of all diagrams which cannot be cut in two pieces by cutting one line), this self-energy is the sum of all diagrams which cannot be cut in two pieces by cutting one line corresponding to site $\mathbf{m}$. That is,

$$\Sigma_{\mathbf{m}}(E) = \sum_{n=1}^\infty \sum_{\mathbf{m}_1,\dots,\mathbf{m}_n\neq\mathbf{m}} \frac{V_{\mathbf{m}\mathbf{m}_n}\dots V_{\mathbf{m}_2\mathbf{m}_1} V_{\mathbf{m}_1\mathbf{m}}}{(E-\epsilon_{\mathbf{m}_n})\dots(E-\epsilon_{\mathbf{m}_2})(E-\epsilon_{\mathbf{m}_1})}. \qquad (1.12)$$

The main object of our study will be $\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta)$ at $\eta\to 0^+$, which determines $iG_{\mathbf{00}}(t)$, the probability amplitude for the particle located at the site $\mathbf{m}=\mathbf{0}$ at $t=0$, to be still found at the same site at later times. Physically, the addition of $i\eta$ corresponds to attaching to every site a sink with the particle loss rate $\eta>0$. For a finite η, the probability to find the particle at the initial site will certainly vanish at $t\to\infty$ at least as $e^{-\eta t}$. The question is whether this property is preserved at $\eta\to 0$. To understand how this is related to the behavior of $\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta)$, it is instructive to analyze two representative cases.

Let us take the first non-vanishing perturbative term,

$$\lim_{\eta\to 0^+} \operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta) = -\lim_{\eta\to 0^+} \sum_{<\mathbf{m}_1>} \frac{V^2\eta}{(E-\epsilon_{\mathbf{m}_1})^2+\eta^2} = -\pi V^2 \sum_{<\mathbf{m}_1>} \delta(E-\epsilon_{\mathbf{m}_1}) \quad (1.13)$$

where the summation is over six sites neigboring the site $\mathbf{m}=0$.

$$\begin{aligned} iG_{\mathbf{00}}(t) &= \lim_{\eta\to 0^+} \int \frac{i\,dE}{2\pi}\, e^{-iEt} \left(E + i\eta - \epsilon_{\mathbf{0}} - \sum_{<\mathbf{m}_1>} \frac{V^2}{E+i\eta-\epsilon_{\mathbf{m}_1}} \right)^{-1} = \\ &= \lim_{\eta\to 0^+} \int \frac{i\,dE}{2\pi} \sum_{j=1}^{7} \frac{C_j e^{-iEt}}{E-E_j+i\eta} = \sum_{j=1}^{7} C_j e^{-iE_j t}. \end{aligned} \quad (1.14)$$

The integrand is a rational function of E, whose denominator is a seventh-degree polynomial. Thus, it has seven poles at $E=E_j$, which are the eigenvalues of the seven-site Hamiltonian, and thus are real. For any $\eta>0$ the poles are shifted in the lower half-plane, and for $t>0$ the integration contour can be closed by an infinite semicircle in the lower half-plane, so the poles are picked up.

Let us now consider $\operatorname{Im}\Sigma_{\mathbf{0}}(E)$ which has a finite limit at $\eta\to 0$, with a support of non-zero measure, for example,

$$\lim_{\eta\to 0^+} \operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta) = -\Gamma\,\theta(E_c-|E|), \quad (1.15)$$

with $|\epsilon_{\mathbf{0}}|<E_c$. In the expression for the Green's function,

$$iG_{\mathbf{00}}(t) = \int_{-\infty}^{\infty} \frac{i\,dE}{2\pi}\, e^{-iEt} \left[E - \epsilon_{\mathbf{0}} - \frac{\Gamma}{\pi} \ln\left|\frac{E+E_c}{E-E_c}\right| + i\Gamma\,\theta(E_c-|E|) \right]^{-1}, \quad (1.16)$$

the integrand has a branch cut between $-E_c$ and E_c, as well as two real poles, $E=E_-<-E_c$ and $E=E_+>E_c$. Thus,

$$\begin{aligned} iG_{\mathbf{00}}(t) &= \int_{-E_c}^{E_c} \frac{\Gamma\,dE}{\pi}\, e^{-iEt} \left[\left(E - \epsilon_{\mathbf{0}} - \frac{\Gamma}{\pi} \ln\frac{E_c+E}{E_c-E} \right)^2 + \Gamma^2 \right]^{-1} + \sum_{\pm} C_\pm e^{-iE_\pm t} \approx \\ &\underset{t\to\infty}{\approx} \frac{2\pi \sin E_c t}{\Gamma t \ln^2 t} + C_+ e^{-iE_+ t} + C_- e^{-iE_- t}. \end{aligned} \quad (1.17)$$

We can conclude the following.

(i) The range of E where $\lim\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta)$ is regular, corresponds to eigenstates of the problem which are extended over the whole sample, so the particle can escape to infinity, while in the range of E where $\lim\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta)$ is zero or infinite, the eigenstates (if any), are localized, so the particle has a finite probability to stay at $\mathbf{m}=\mathbf{0}$.

(ii) In any finite order of the perturbation theory, $\lim\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta)$ is a sum of a finite number of δ-functions, so a regular limit can be obtained only by summing the series to infinite order first, and sending $\eta\to 0$ afterwards.

(iii) The distinction between the two types of behavior of $\lim\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta)$ is completely lost if one averages over the disorder. Indeed,

$$\langle\delta(E-\epsilon_{\mathbf{m}_1})\rangle = \frac{1}{W}\,\theta(W/2-|E|). \quad (1.18)$$

Thus, $\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta) \equiv -\Gamma$ being a random variable because it depends on all $\{\epsilon_{\mathbf{m}}\}$, we have to analyze the whole distribution function $P_{E,\eta}(\Gamma)$. In the region of E, where $\lim \operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta)$ is finite, its distribution function is regular at $\eta \to 0$. In the other region, $\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta)$ is a sum of Lorentzians of the width η and height V^2/η at random positions spaced by W. Thus, with probability ~ 1, E falls between two Lorentizans, where $\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta) \sim \eta V^2/W^2$. However, with a small probability $\sim \eta/W$, E hits a peak, and then $\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta) \sim V^2/\eta$. Thus, $P_{E,\eta}(\Gamma)$ has a peak $O(1/\eta)$ at $\Gamma = O(\eta)$, and a long tail which goes to $\Gamma = O(1/\eta)$, where it is $O(\eta^2)$. Matching these two conditions by a power-law tail, we obtain

$$P_\eta(\Gamma) \sim \frac{C}{\Gamma^\alpha}, \quad P_\eta(\eta V^2/W^2) \sim \frac{W^2}{\eta V^2}, \quad P_\eta(V^2/\eta) \sim \frac{\eta}{W}\frac{\eta}{V^2} \quad \Rightarrow \quad P_\eta(\Gamma) \sim \frac{V}{W}\sqrt{\frac{\eta}{\Gamma^{3/2}}}. \tag{1.19}$$

Thus, the general criterion to distinguish between localized and extended states is

$$\lim_{\eta\to 0^+} P_{E,\eta}(\Gamma) = \begin{cases} \text{finite} & \Rightarrow \quad \text{extended,} \\ 0 & \Rightarrow \quad \text{localized.} \end{cases} \tag{1.20}$$

In the imaginary part of the product of n complex numbers,

$$\operatorname{Im}[(x_1+iy_1)\dots(x_n+iy_n)] = y_1x_2\dots x_n + x_1y_2x_3\dots x_n + \dots + x_1\dots x_{n-1}y_n + \\ + y_1y_2y_3x_4\dots x_n + \dots,$$

we pick up the contribution with the imaginary part of each single factor, since other terms will be at least η^3:

$$\operatorname{Im}\prod_{j=1}^{n}\frac{1}{E-\epsilon_{\mathbf{m}_j}+i\eta} = \sum_{k=1}^{n}\frac{-\eta}{(E-\epsilon_{\mathbf{m}_k})^2+\eta^2}\prod_{j(\neq k)}\frac{1}{E-\epsilon_{\mathbf{m}_j}} + O(\eta^3). \tag{1.21}$$

If we are not interested in the cutting off the power-law tail of $P(\Gamma)$ at $\Gamma \sim 1/\eta$, we can neglect η^2 in the denominator, so that

$$\operatorname{Im}\Sigma_{\mathbf{0}}(E+i\eta) = -\sum_{\mathbf{m}}\frac{\eta V^2}{(E-\epsilon_{\mathbf{m}})^2}\left|\sum_{\text{paths:}\mathbf{0}\to\mathbf{m}}\prod_{\mathbf{m}_j\in\text{path}}\frac{V}{E-\epsilon_{\mathbf{m}_j}}\right|^2 + O(\eta^3) = \\ = -\eta\left|\sum_{\text{paths}}A_{\text{path}}\right|^2, \quad A_{\text{path}} \equiv \prod_{\mathbf{m}_j\in\text{path}}\frac{V}{E-\epsilon_{\mathbf{m}_j}} \tag{1.22}$$

where the last sum is over all paths which start from the site $\mathbf{0}$ and never return to it. If $\Gamma = \eta A^2$ and $P(\Gamma) \propto \sqrt{\eta/\Gamma^3}$, then $P(A) \propto 1/A^2$. Since A does not contain η, our only hope to obtain some finite Γ at $\eta \to 0$ is to make explode the distribution of A at $n \to \infty$. Now we have to analyze the probability distribution of the amplitude A for each path, and of their sum for all paths.

Let us fix $E = 0$ for simplicity (the center of the band). Let us first consider non-self-intersecting paths of a given length n. Then all factors in A are statistically independent, so the probability distribution

$$P_n(A) = \int\limits_{-W/2}^{W/2} \frac{d\epsilon_1}{W} \cdots \frac{d\epsilon_n}{W}\, \delta\Big(A - \frac{V}{\epsilon_1} \cdots \frac{V}{\epsilon_n}\Big). \tag{1.23}$$

The sign is random, so $P_n(A) = P_n(-A) = (1/2)P(|A|)$. Thus, we can write

$$P_n(A > 0) = \frac{1}{2} \int\limits_{0}^{W/2} \frac{d\epsilon_1}{W/2} \cdots \frac{d\epsilon_n}{W/2}\, \delta\Big(A - \frac{V}{\epsilon_1} \cdots \frac{V}{\epsilon_n}\Big). \tag{1.24}$$

Let us change the integration variables $\epsilon_j = (W/2)e^{-\lambda_j}$ and denote $v \equiv 2V/W$. Then,

$$P_n(A > 0) = \frac{1}{2} \int\limits_{0}^{\infty} e^{-(\lambda_1 + \lambda_n)} d\lambda_1 \ldots d\lambda_n\, \delta\Big(A - v^n e^{\lambda_1 + \ldots + \lambda_n}\Big). \tag{1.25}$$

Using the fact that

$$\delta\left(A - v^n e^{\mathcal{L}}\right) = \frac{1}{v^n e^{\mathcal{L}}}\, \delta\Big(\mathcal{L} - \ln \frac{A}{v^n}\Big), \tag{1.26}$$

we obtain

$$\begin{aligned} P_n(A > 0) &= \frac{v^n}{2A^2} \int\limits_{0}^{\infty} d\lambda_1 \ldots d\lambda_n\, \delta\Big(\lambda_1 + \ldots + \lambda_n - \ln \frac{A}{v^n}\Big) = \\ &= \frac{v^n}{2A^2}\, \theta(A - v^n) \ln^{n-1} \frac{A}{v^n} \int\limits_{0}^{\infty} d\lambda_1 \ldots d\lambda_n\, \delta(\lambda_1 + \ldots + \lambda_n - 1). \end{aligned} \tag{1.27}$$

The last integral is evaluated as

$$\int\limits_{0}^{\infty} d\lambda_1 \ldots d\lambda_n \int\limits_{-\infty}^{\infty} \frac{ds}{2\pi}\, e^{(is - 0^+)(\lambda_1 + \ldots + \lambda_n - 1)} = \int\limits_{-\infty}^{\infty} \frac{ds}{2\pi} \frac{e^{-is}}{(-is + 0^+)^n} = \frac{1}{(n-1)!}. \tag{1.28}$$

This gives an exact expression

$$P_n(A > 0) = \frac{v^n}{2A^2} \frac{\theta(A - v^n)}{(n-1)!} \ln^{n-1} \frac{A}{v^n} \approx \frac{(ev)^n \theta(A - v^n)}{\sqrt{8\pi n} A^2} \left(\ln \frac{1}{v} + \frac{\ln A}{n}\right)^{n-1}, \tag{1.29}$$

which is nothing but the Poisson distribution for $\ln(A/v^n)$.

Now we have to sum over the amplitude over N different paths, $A = A_1 + \ldots + A_N$. For any normal lattice, the number of non-self-intersecting paths of length n scales as $N = C_n K^n$, where C_n is a power-law prefactor (which is not important for us, as we will see shortly), and K is called connectivity of the lattice. For the 3D cubic lattice, $K \approx 4.5$.

Let us neglect the statistical correlations between different amplitudes. Then we need to find the distribution of the sum of a large number of independent random variables. Normally, this is done by using the central limit theorem. However, the distribution $P_n(A)$ does not satisfy the conditions of the central limit theorem, since both $\langle A \rangle$ and $\langle A^2 \rangle$ diverge. This happens because of the slowly decreasing tail:

$$P_{n \gg 1}(A) \propto \begin{cases} 1/A^{2 - 1/\ln(1/v)}, & v^n < A \ll 1/v^n, \\ \ln^n A / A^2, & A \gg 1/v^n. \end{cases} \tag{1.30}$$

For such distributions, where large values of A occur too easily, the whole sum is dominated by its largest term, so the probability distribution of the sum coincides with that of the largest term:

$$P_{N,n}(A) \approx N P_n(A) \left[1 - \int_A^\infty P_n(A')\,dA'\right]^{N-1} \approx \frac{d}{dA} \exp\left[-N \int_A^\infty P_n(A')\,dA'\right]. \qquad (1.31)$$

Let us calculate the integral in the exponent:

$$\int_A^\infty \frac{v^n}{2(A')^2} \frac{\ln^{n-1}(A'/v^n)}{(n-1)!}\,dA' \underset{A'=Ae^\lambda}{=} \int_0^\infty \frac{v^n e^{-\lambda} d\lambda}{2} \sum_{k=0}^{n-1} \frac{\lambda^{n-1-k} \ln^k(A/v^n)}{(n-1-k)!k!} = \frac{v^n}{2A} \sum_{k=0}^{n-1} \frac{\ln^k(A/v^n)}{k!}. \qquad (1.32)$$

At $n \to \infty$ the sum is not equal $e^{\ln(A/v^n)}$; in fact, for $v < 1$ the sum is divergent. For $k = n-1-l$, $l \ll n$, it can be approximated by a geometric series:

$$\frac{\ln^{n-1-l}(A/v^n)}{(n-1-l)!} \approx \frac{1}{\sqrt{2\pi n}} \left[\frac{e \ln(A/v^n)}{n-1}\right]^{n-1-l} \frac{1}{[1 - l/(n-1)]^{n-1-l}} \approx$$
$$\approx \frac{e^n}{\sqrt{2\pi n}} \left[\ln\frac{1}{v} + \frac{\ln A}{n}\right]^{n-1-l} \approx \frac{e^n \ln^{n-1}(1/v)}{\sqrt{2\pi n}} \frac{A^{1/\ln(1/v)}}{\ln^l(1/v)}, \quad \ln A \ll n \ln\frac{1}{v}.$$

Assuming $v < 1/e$, we can sum the geometric series, and obtain

$$P_{N,n}(A) \approx \frac{d}{dA} \exp\left\{-C_n' \frac{[evK\ln(1/v)]^n}{A^{1-1/\ln(1/v)}}\right\} =$$
$$= C_n'' \frac{[evK\ln(1/v)]^n}{A^{2-1/\ln(1/v)}} \exp\left\{-C_n' \frac{[evK\ln(1/v)]^n}{A^{1-1/\ln(1/v)}}\right\}. \qquad (1.33)$$

At $n \to \infty$ this distribution shrinks to the left (smaller values of A) if $evK\ln(1/v) < 1$, while in the opposite case it explodes to the right. Thus, the critical value of v is determined by the equation

$$eKv_c \ln\frac{1}{v_c} = 1. \qquad (1.34)$$

It is indeed smaller than $1/e$. So, the perturbation converges and the state remains localized if $v < v_c$. Note that the transition point is determined by the competition of different factors, exponential in n, so power-law prefactors do not matter.

Let us now allow self-intersecting paths. The danger to the previous arguments come from some rare events. For example, suppose that for some path it happened that $\epsilon_{\mathbf{m}_j} \approx E(=0)$. Then one can build paths which return to this point many times. The contribution from such paths can be summed to all orders, resulting in the replacement

$$E - \epsilon_{\mathbf{m}_j} \to E - \epsilon_{\mathbf{m}_j} - \Sigma_{\mathbf{m}_j}(E).$$

If $|\Sigma_{\mathbf{m}_j}(E)| < W/2$, it results in a random correction to a random quantity $\epsilon_{\mathbf{m}_j}$, which does not affect the result. However, $\Sigma_{\mathbf{m}_j}(E)$ can become large if occasionally $\epsilon_{\mathbf{m}_i} \approx \epsilon_{\mathbf{m}_{i+1}} \approx E$. Then the denominator $E - \epsilon_{\mathbf{m}_j} - \Sigma_{\mathbf{m}_j}(E)$ cannot vanish. To take this effect into account, Anderson proposed to modify the probability distribution of $E - \epsilon_{\mathbf{m}}$: instead of using the simple box one, we put the probability to zero for $|E-\epsilon_{\mathbf{m}}| < \Delta/2$. The quantity Δ should be such that $V^2/|E-\epsilon_{\mathbf{m}}| > W/2$ are prohibited, that is, $\Delta = 4V^2/W$. Let us write

$$A = (-1)^n \frac{V}{\epsilon_1} \dots \frac{V}{\epsilon_n} = \pm v^n e^{\mathcal{L}}, \quad 0 < \mathcal{L} < n \ln \frac{W}{\Delta}, \tag{1.35}$$

$$P_n(A>0) = \frac{1}{2} \int_0^\infty \delta\left(A - v^n e^{\mathcal{L}}\right) \mathcal{P}_n(\mathcal{L})\, d\mathcal{L} = \frac{\mathcal{P}_n(\ln(A/v^n))}{2A}, \tag{1.36}$$

and calculate the distribution of $\mathcal{L}$ via its characteristic function

$$\chi_n(s) = \left\langle e^{-s\mathcal{L}} \right\rangle = \int_{\Delta/W}^{1} \frac{dx_1}{1-\Delta/W} \cdots \frac{dx_n}{1-\Delta/W} \exp\left(-s \ln\frac{1}{x_1} - \dots - s\ln\frac{1}{x_n}\right) =$$

$$= \left(\int_{\Delta/W}^{1} \frac{dx\, x^s}{1-\Delta/W} \right)^n = \left[\frac{W}{W-\Delta} \, \frac{1-(\Delta/W)^{s+1}}{s+1} \right]^n . \tag{1.37}$$

The Laplace transform $\bar{f}(s)$ of a function $f(t)$, such that $f(t<0)=0$ is defined as

$$\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \tilde{f}(is), \tag{1.38}$$

where $\tilde{f}(\omega)$ is the Fourier transform, analytical for $\mathrm{Im}\,\omega > 0$ or for $\mathrm{Re}\, s > 0$. The inverse transform is calculated as

$$f(t) = \int_{-\infty}^{\infty} \bar{f}(-i\omega)\, e^{-i\omega t} \frac{d\omega}{2\pi} = \int_{-i\infty}^{i\infty} \bar{f}(s)\, e^{st} \frac{ds}{2\pi i}, \tag{1.39}$$

where the integration contour goes vertically anywhere in the right half-plane of s.

$$\mathcal{P}_n(\mathcal{L}) = \int_{-i\infty}^{i\infty} \chi_n(s)\, e^{s\mathcal{L}} \frac{ds}{2\pi i} =$$

$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \exp\left\{ s\mathcal{L} + n \ln \frac{W}{W-\Delta} - n\ln(s+1) + n \ln\left[1 - \left(\frac{\Delta}{W}\right)^{s+1}\right] \right\}. \tag{1.40}$$

$\chi_n(s)$ is analytical everywhere, even at $s=-1$ which is not a pole. Indeed,

$$\frac{1-a^z}{z} = \frac{1 - z\ln a + O(z^2)}{z} = -\ln a + O(z).$$

However, $\chi_n(s)$ grows for $\mathrm{Re}\, s \to -\infty$ since $\Delta/W < 1$. At $\mathrm{Re}\, s \to +\infty$, e^{st} in the inverse Laplace transform is growing. Thus, we cannot close the contour at infinity, but we can deform it arbitrarily. In particular, we can deform it to the stationary point of the exponent and calculate the integral by steepest descent, valid at $n \gg 1$. The equation for the stationary point s_* is

$$\frac{\mathcal{L}}{n} = -\frac{\partial \ln \chi_1(s_*)}{\partial s_*} = \frac{1}{s_*+1} - \frac{\ln(W/\Delta)}{(W/\Delta)^{s_*+1} - 1}. \tag{1.41}$$

At $s_* = -1$, the right-hand side is $\ln\sqrt{W/\Delta}$. For $s_* > -1$, it decreases as $1/(s_*+1)$, the second term is exponentially suppressed. For $s_* \to -\infty$, one can neglect $(W/\Delta)^{s_*+1}$, so the right-hand side asymptotically approaches $\ln(W/\Delta)$, the largest allowed value of $\mathcal{L}/n$. The distribution function can be approximated by $\mathcal{P}_n(\mathcal{L}) \sim e^{s_*\mathcal{L}}\chi_n(s_*)$ (the prefactors are not important, as we are struggling for exponentials):

$$\mathcal{L} \ll \frac{n}{2}\ln\frac{W}{\Delta}: \qquad \mathcal{P}_n(\mathcal{L}) \sim e^{-\mathcal{L}}\left[\frac{eW\mathcal{L}/n}{W-\Delta}\right]^n, \tag{1.42}$$

$$\mathcal{L} = \frac{n}{2}\ln\frac{W}{\Delta}: \qquad \mathcal{P}_n(\mathcal{L}) \sim e^{-\mathcal{L}}\left[\frac{W\ln(W/\Delta)}{W-\Delta}\right]^n, \tag{1.43}$$

$$\mathcal{L} \to n\ln\frac{W}{\Delta}: \qquad \mathcal{P}_n(\mathcal{L}) \sim e^{-\mathcal{L}}\left[\frac{eW}{W-\Delta}\left(\ln\frac{W}{\Delta}-\frac{\mathcal{L}}{n}\right)\right]^n, \tag{1.44}$$

where the last expression is obtained by writing the exponent as

$$n\ln\frac{W}{W-\Delta} - \mathcal{L} - n\ln|s_*+1| + n|s_*+1|\left(\ln\frac{W}{\Delta}-\frac{\mathcal{L}}{n}\right) + n\ln\left[1-\left(\frac{\Delta}{W}\right)^{|s_*+1|}\right],$$

and neglecting all corrections $\propto (\Delta/W)^{|s_*+1|}$. Thus, we can write

$$P_n(A) \sim \frac{1}{A^2}\left[\frac{ev}{1-\Delta/W}\,\psi\left(\ln\frac{1}{v}+\frac{\ln A}{n}\right)\right]^n, \tag{1.45}$$

where the function $\psi(x)$ is defined as

$$\psi(x) = x(s_*+1) + \ln\chi_1(s_*) - \ln\frac{eW}{W-\Delta}, \tag{1.46}$$

and s_* is found from Eq. (1.41) with $\mathcal{L}/n \to x$. Above we have determined the asymptotic behaviour of $\psi(x)$:

$$\psi(x\to 0) \approx x, \quad \psi\left(\ln\sqrt{W/\Delta}\right) = \frac{1}{e}\ln\frac{W}{\Delta}, \quad \psi(x\to\ln(W/\Delta)) \approx \ln\frac{W}{\Delta} - x. \tag{1.47}$$

Eq. (1.41) implies that $\psi(x)$ reaches the maximum at $x = \ln\sqrt{W/\Delta}$, since

$$\frac{d\psi}{dx} = s_* + 1 + \left(x + \frac{\partial\ln\chi_1}{\partial s_*}\right)\frac{\partial s_*}{\partial x}. \tag{1.48}$$

To determine the critical point, we need to take the maximum value of ψ, which is $(1/e)\ln(W/\Delta)$. Summation over the paths gives the factor K^n, as before. Using $\Delta = 4V^2/W$, we obtain the modified equation for the critical point:

$$\frac{2Kv_c\ln(1/v_c)}{1-v_c^2} = 1. \tag{1.49}$$

For $K = 4.5$, this gives $v_c \approx 0.032$. Note that Δ has enhanced the localization.

As we have seen, the arguments given by Anderson in 1958 were not fully rigorous. For example, statistical correlation between different amplitudes were neglected. The proper resummation of loops was replaced by a simple cutoff Δ. In some situations these assumptions are indeed wrong (as we will see later). So in the beginning, very few people took this seriosly. However, in 1973, Abou-Chacra, Anderson and Thouless gave an exact solution of the problem on the Bethe lattice, which agreed very well with the 1958 paper. Later, numerical studies have been performed, and gave for the 3D cubic lattice $v_c \approx 0.12$. The full theory of Anderson transition in 3D is still missing.